THE AUSTRALIAN POST OFFICE

CP 308

PRICE 20c

COURSE OF TECHNICAL INSTRUCTION

Engineering Training Section, Headquarters, Postmaster-General's Department, Melbourne

# INTRODUCTION TO PULSE TECHNIQUES

Page



★ AMENDED 1965

## 1. INTRODUCTION.

1.1 The first electrical communication scheme transferred information from one place to another in coded form, with the information being represented by groups of short and long bursts, or "pulses", of electrical energy. In modern electronics, pulses are finding more and more applications, as their original application in telegraphy for message transmission has been streamlined and extended to many other fields. A knowledge of pulses is required for an understanding of television, radar, electronic computers and the transmission of computer data, and for methods of communication using pulse modulation.

1.2 To successfully study pulse circuits, it is necessary to understand the terms used. This paper defines the types of pulses commonly encountered and the methods used to specify the characteristics of pulses.

Complex waves consist of a fundamental and many harmonics. So that the bandwidth necessary to transmit a pulse can be determined, the harmonics required to accurately reproduce a pulse with specified characteristics are examined.

The explanations of the operation of pulse circuits are based on the "charge" and "discharge" of capacitors and inductors through resistors. With this in mind the currents and voltages existing in resistor-capacitor and inductor-resistor circuits are revised. These circuits are considered in Applied Electricity II in the paper "Waveforms, Timing and Oscillatory Circuits", but the further discussion here shows how accurate calculations can be made to find the state of charge at any time.

## 2. DEFINITIONS.

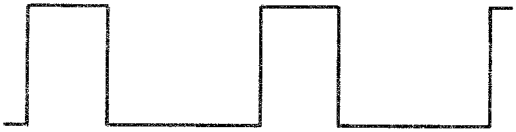
2.1 Wave Shape. The initial description of any wave is by specifying its shape. The most common waveform considered is a sine wave, which is a graph of the sine of the angle of rotation of a vector as the angle varies with time. Other types of waveforms often encountered are -

(i) "Rectangular Waves" - Waves that alternate between two fixed values with negligible transition time compared with the duration of a cycle of the wave (Fig. 1a).

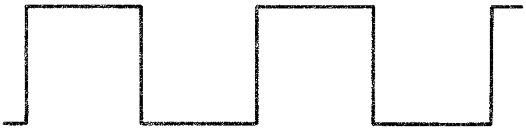
(ii) "Square Waves" - Special cases of rectangular waves where equal lengths of time are spent at each of the two fixed values (Fig. 1b).

(iii) "Sawtooth Waves" - Waves with a linear rate of change from one value to another, followed by a return or "retrace" to the initial value with negligible transition time compared with the duration of a cycle (Fig. 1c).

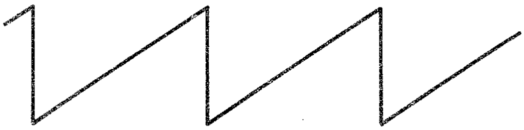
A "pulse" is a sudden change of voltage or current of short duration compared with the time scale of interest, with the voltage or current having the same value both before and after the pulse. Pulses are also described according to their shape, so we have rectangular pulses, and sawtooth pulses (Fig. 2).



(a) Rectangular Wave.



(b) Square Wave.

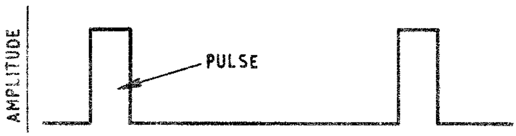


(c) Sawtooth Wave.

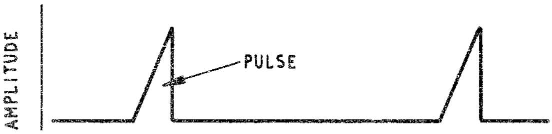
FIG. 1. WAVEFORMS.

2.2 Polarity. The pulses may be either the sections of the waveform extending positive or extending negative from the normal steady value between pulses and these sections are described as positive and negative pulses respectively. This still applies if the pulse waveform is superimposed on a D.C. signal as in Fig. 2d. Positive pulses are shown in Figs. 2a and b, and negative pulses in Figs. 2c and d.

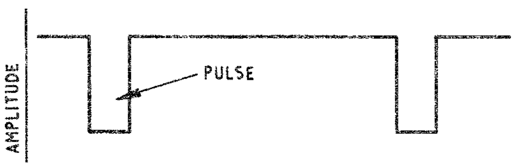
A positive "pulse train" or sequence of pulses, is changed to a negative pulse train by a phase reversal. This can be produced by a transformer or by a valve amplifier without changing the shape of the waveform except to invert it. The pulse waveform in Fig. 2a is changed to the waveform in Fig. 2c by a phase reversal.



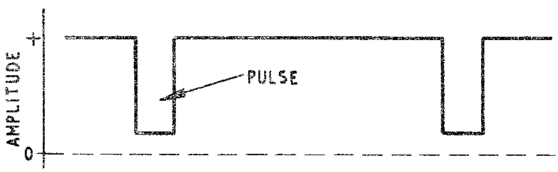

(a) Positive Rectangular Pulses.




(b) Sawtooth Pulses.




(c) Negative Rectangular Pulses.




(d) Negative Pulses on Positive D.C. Signal.

FIG. 2. PULSES.

2.3 Transition Time. Practical waveforms do not have "transitions" (changes) that are instantaneous because of the limiting of the high frequency response. Section 3 discusses how the time for the transition depends on the amplitude of the high frequency components that make up the wave. A practical rectangular wave is often shaped as shown in Fig. 3, where the pulse has sloping sides. This wave shape is sometimes described as being "trapezoidal".

The time for the transition of a rectangular wave (which includes a rectangular pulse waveform), is an important characteristic. The transition time is specified by either the "rise time" or the "decay time".

(i) "Rise Time" is the time taken for a waveform to vary between 10% and 90% of the final peak to peak amplitude of the transition, unless some other levels are stated. This is designated $t_r$ in Fig. 3. The 10% and 90% points are specified as these times are more easily determined than the times where the wave departs from 0 and just reaches 100%.

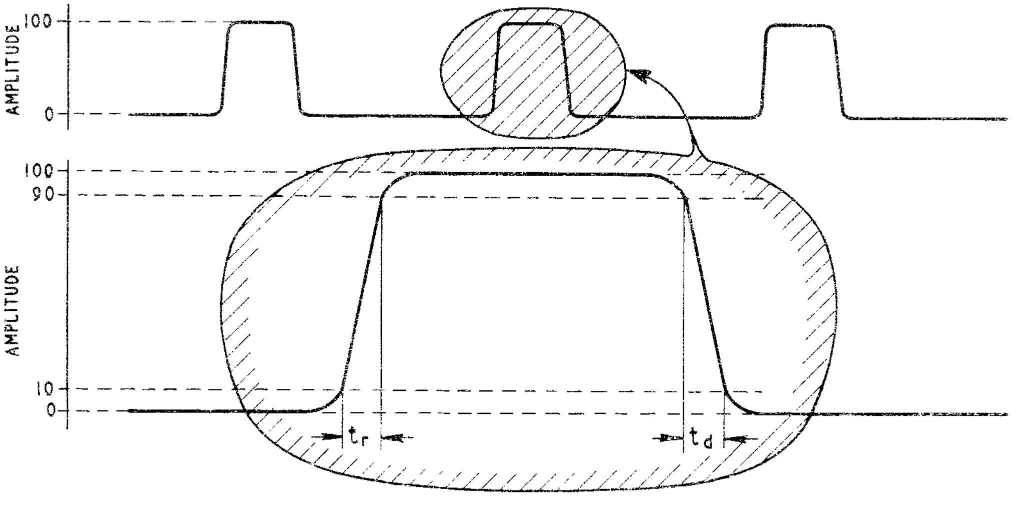



FIG. 3. TRANSITIONS OF A RECTANGULAR WAVE.

(ii) "Decay Time" is the time taken for a waveform to vary between 90% and 10% of the peak to peak amplitude unless otherwise stated, ($t_d$ in Fig. 3). The decay time is often equal to the rise time, but this is not necessarily so.

In many cases the term "decay time" is not used, and instead we speak of the rise time of the leading edge of the pulse and the rise time of the trailing edge of the pulse.

With sawtooth waves, where the linear transition is the major section of the wave and its limits are normally well defined, the transition times specified are the time for the forward trace and the time for retrace, as designated in Fig. 4.

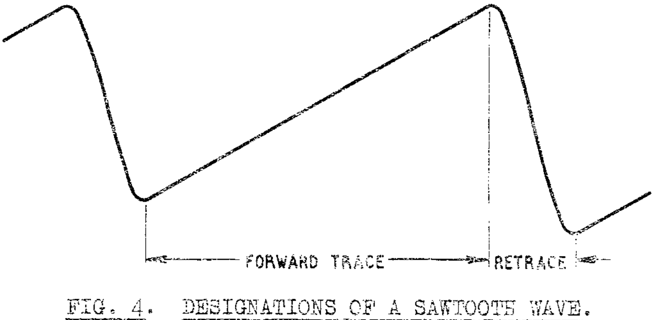



FIG. 4. DESIGNATIONS OF A SAWTOOTH WAVE.

2.4 "Pulse Duration" is the time for the pulse measured at specified points on the transition. A number of points are commonly chosen. For rectangular pulses, pulse duration is normally measured as the interval between the end of the rise time and the start of the decay time, i.e. the time for the pulse measured at approximately the maximum amplitude ($t_1$ in Fig. 5). In some cases, however, the duration is specified from the start of the rise time to the end of the decay time with the transition times considered as part of the pulse duration.

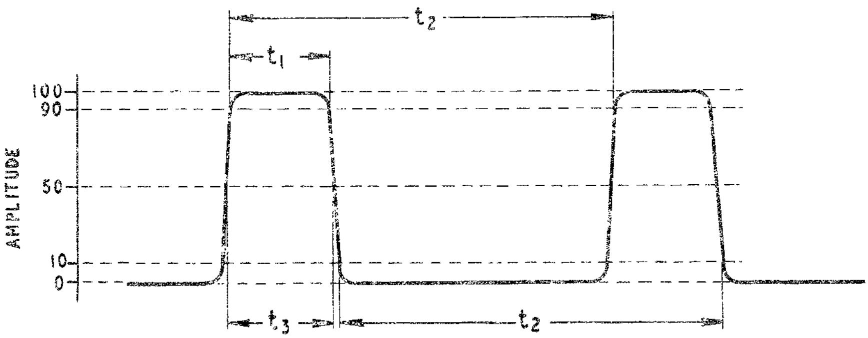



FIG. 5. PULSE CHARACTERISTICS.

Another point used for measurement of duration is at half the pulse peak-to-peak amplitude ($t_3$ in Fig. 5). This time is the pulse "half amplitude duration" (h.a.d.). Specification of the pulse h.a.d. is sometimes used for rectangular pulses, but is more useful for measuring the duration of pulses where the transition time of the pulse is almost the same as the pulse duration. An example of this designation in practice is for specifying the duration of the "sine squared pulse" considered in the paper "Video Test Signals".

2.5 "Pulse Spacing" (sometimes known as "Pulse Period" or "Pulse Repetition Time" (P.R.T.)), is the time between corresponding points on two consecutive pulses ($t_2$ in Fig. 5).

The same value is obtained independent of which point is taken.

Related to the pulse spacing is the "Pulse Repetition Frequency" (P.R.F.), which is the number of times the pulse re-occurs per second, i.e.

$$\text{P.R.F.} = \frac{1}{\text{Pulse Spacing}} = \frac{1}{t_2}.$$

2.6 Combining the pulse spacing and the pulse duration we have the "Pulse Duty Factor". The "Pulse Duty Factor" is the ratio of the pulse duration to the pulse spacing, i.e.

$$\text{Pulse Duty Factor} = \frac{\text{Pulse Duration}}{\text{Pulse Spacing}} = \frac{t_1}{t_2}.$$

Notice that this is also equal to the pulse duration times the pulse repetition frequency.

$$\text{Pulse Duty Factor} = \text{Pulse Duration} \times \text{P.R.F., since}$$

$$\frac{1}{\text{Pulse Spacing}} = \text{P.R.F.}$$

Sometimes "Mark-to-Space Ratio" is used to define the ratio of pulse duration to the time from the conclusion of one pulse to the start of the next. Therefore, in Fig. 5,

$$\text{Mark-to-Space Ratio} = \frac{t_1}{t_2 - t_1}.$$

The preceding definitions have been applied to rectangular pulses, but where applicable are also used to describe other waveforms.

## 3. FREQUENCY SPECTRUM.

★ 3.1 Square Waves. Complex waves can be considered as being formed by the algebraic sum of a large number of component sine waves. One way of examining a circuit connected with pulses is to consider the component frequencies of the wave, and then examine the effects of the amplitude-frequency and phase-frequency responses of the circuit, on these component frequencies.

In the paper "Waveforms, Timing and Oscillatory Circuits" of Applied Electricity 2, it is shown that a square wave is composed of a fundamental and a number of odd harmonics. All odd harmonics to infinity must be added in the correct phase and with the correct amplitude, to construct a perfect square wave, although in practice the higher order harmonics can be neglected as they become small and insignificant in amplitude.

To obtain the desired result, components must all be zero and increasing at the same time (A in Fig. 6a) and the amplitudes must be according to the reciprocal of the order of the harmonic. That is, a square wave is composed of:

- the fundamental,
- the third harmonic with 1/3 the amplitude of the fundamental,
- the fifth harmonic with 1/5 the amplitude of the fundamental,
- the seventh harmonic with 1/7 the amplitude of the fundamental, and so on.

Fig. 6a shows the fundamental and the phase and amplitude relationships of the 3rd and 5th harmonics. The sum of the required harmonics up to the 5th, the 13th and the 21st is shown in Figs. 6b, c and d respectively.

As the higher frequencies are added the rise time of the transitions decreases and the number of ripples in the resultant increases. The frequency of the ripples or "ringing" is the same as the frequency of the highest order harmonic included. Adding further harmonics causes the ringing to be confined more to the sections of the waveform adjacent to the transitions. The relationship between the highest frequency present and the rise time of the resultant waveform is given in para. 3.6.

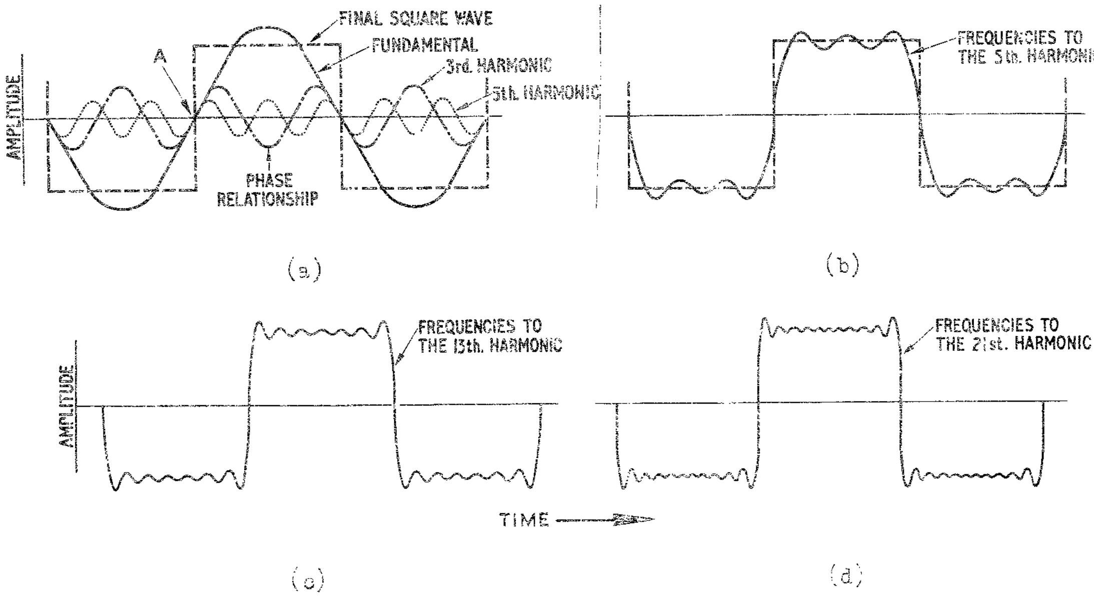



FIG. 6. COMPONENTS OF A SQUARE WAVE.

It is convenient to show the frequencies contained by a complex waveform by plotting the magnitude of the components on a frequency scale as shown in Fig. 7 for the square wave.

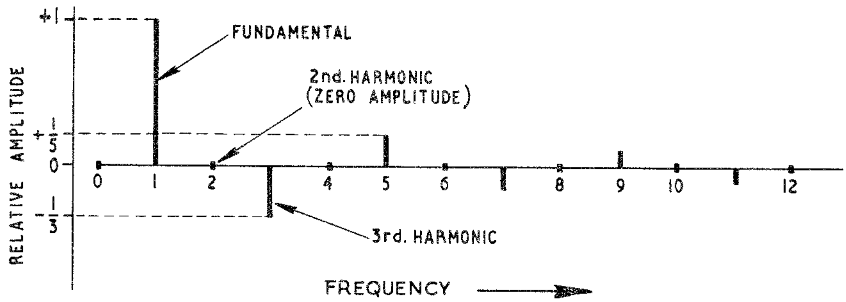



FIG. 7. FREQUENCY SPECTRUM OF A SQUARE WAVE.

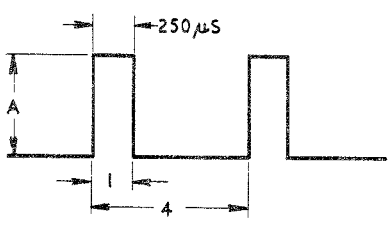
The fundamental and the 5th harmonic are drawn with a positive sign and the 3rd and 7th harmonics with a negative sign to indicate their phase relationship at a convenient reference time. The phase relationship is confirmed by examining Fig. 6, which shows that, at the time the fundamental is at a positive maximum the 5th harmonic is also at a positive maximum, and the 3rd harmonic is at a negative maximum. This means that at the reference time alternate harmonics are of opposite phase. The resultant voltage at this instant is equal to the algebraic sum of the peak amplitudes of the components, that is Fundamental minus 3rd harmonic plus 5th harmonic.

3.2 Short Duration Pulses. For television and many other applications, a major interest is in rectangular pulses with a small pulse duty factor. Consider a pulse train in which the pulse repetition frequency is 1000c/s and the pulse duty factor is 1/4. In this case, the pulse duration is 250µS (Fig. 8a).

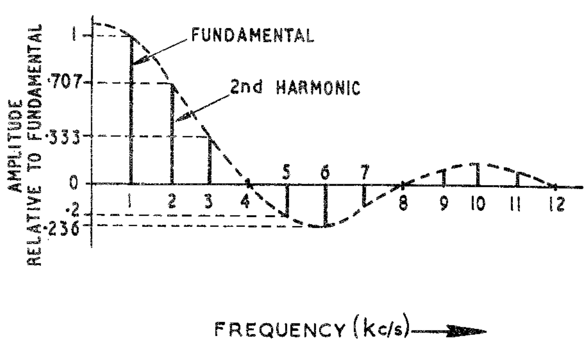
The frequencies required to produce such a pulse are tabulated in Table 1 and shown graphically in Fig. 8b.

| Frequency | Amplitude relative to Amplitude of Fundamental |
|---|---|
| Fundamental (1000c/s) | 1 |
| 2nd Harmonic (2000c/s) | .707 |
| 3rd Harmonic (3000c/s) | .333 |
| 4th Harmonic (4000c/s) | No Amplitude |
| 5th Harmonic (5000c/s) | .2 |
| 6th Harmonic (6000c/s) | .236 |
| 7th Harmonic (7000c/s) | .143 |
| 8th Harmonic (8000c/s) | No Amplitude |
| etc. | |

TABLE 1. FREQUENCY COMPONENTS OF A PULSE.




(a)




(b)

FIG. 8. FREQUENCY SPECTRUM OF A PULSE.

By joining the amplitudes of all the components shown for the pulse on the frequency scale (Fig. 8b), the values are seen to follow the shape of a "damped sine wave" with zero amplitudes for the components at regular intervals of frequency. The first zero is at the harmonic corresponding to $\frac{1}{\text{pulse duty factor}}$, which is at a frequency corresponding to $\frac{1}{\text{pulse duration}}$. Further zeros occur at multiples of this first frequency.

In the case with a duty factor of 1/4, the 4th harmonic and the 8th, 12th and 16th, etc., have no amplitude. Therefore, with a pulse duration of 250µS, the first zero is at a frequency of $\frac{10^6}{250}$ c/s, or 4000c/s.

As well as the sine wave components there may be a D.C. component required to construct the pulse train. This can also be included in the drawing of the pulse frequency spectrum by showing a component at zero frequency. The D.C. component magnitude depends on the position of the zero axis of the pulse train and is considered in Section 4.

The fundamental and the harmonics up to the fifth are plotted in Fig. 9a. Note that the fifth harmonic reaches a negative maximum at the same time as the fundamental is a positive maximum as indicated by the phase reversal of this component in the frequency spectrum in Fig. 8b.

The resultant waveform in Fig. 9b is produced by adding all components up to the 8th harmonic. The higher order components are not included in Fig. 9a, as they would confuse the diagram. The resultant shows a resemblance to the desired pulse even with this small number of components.

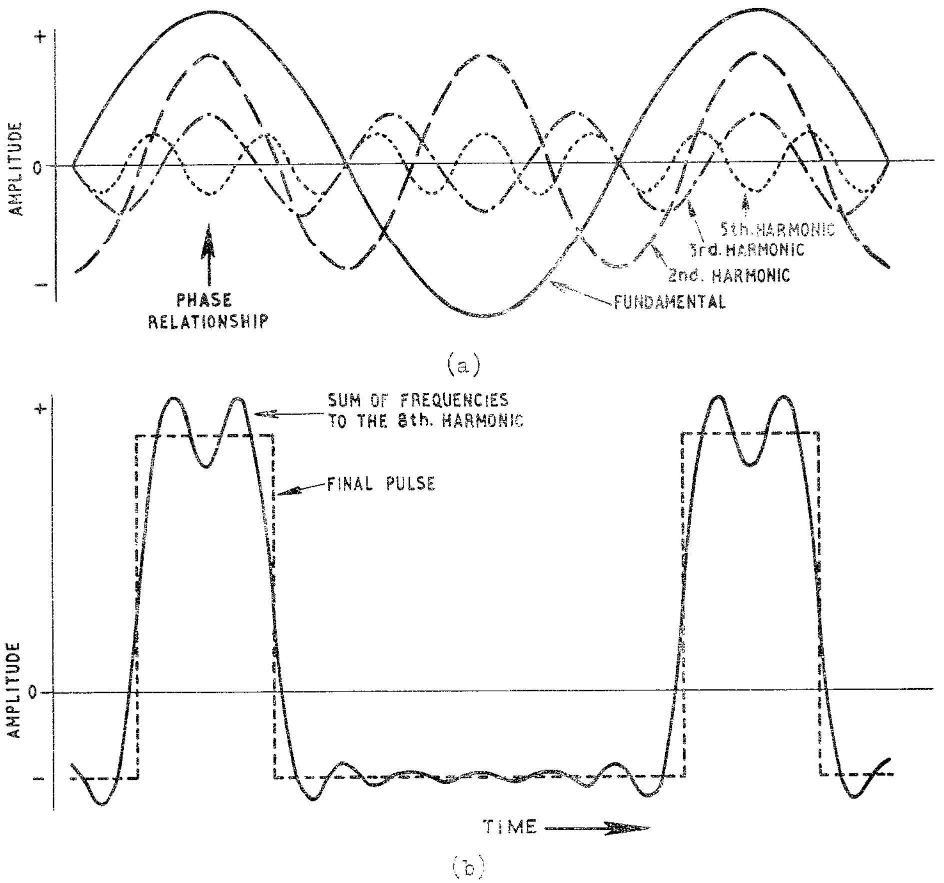



FIG. 9. COMPONENTS OF A PULSE.

3.3 Extending the observations of component frequency amplitudes of waveforms to still shorter duration rectangular pulses, a 2µS pulse with a P.R.F. of 1000c/s has a spectrum of components as in Fig. 10a, where the 500th harmonic, i.e. the 0.5Mc/s component, has zero amplitude, and there are components every 1000c/s representing each of the harmonics. The harmonics have amplitudes as indicated by the figure, and these amplitudes have been calculated and expressed relative to the amplitude of the pulse.

A 1µS pulse with a P.R.F. of 1000c/s has a spectrum as in Fig. 10b, with the 1000th harmonic (1Mc/s component) of zero amplitude, and components at 1000c/s intervals.

Comparing the two spectrums shows that for the same pulse amplitude and pulse repetition frequency, the first zero for the 2µS pulse occurs at half the frequency of the first zero for the 1µS pulse, but the low frequency components of the 2µS pulse have twice the amplitude of the corresponding components of the 1µS pulse.

A further comparison of two pulse spectrums in Fig. 10b and c with a 1µS pulse in each case but with a 1000c/s P.R.F. in one case and a 500c/s P.R.F. in the other case, shows that as the formula stated (frequency of 1st zero $= \frac{1}{\text{pulse duration}}$) the first zero is independent of the pulse repetition frequency and is only dependent on the pulse duration. The components are spaced 500c/s apart in the lower P.R.F. case compared with 1000c/s in the higher P.R.F. case, and the amplitude of the low frequency components in the 500c/s case is half the amplitude of the corresponding components in the 1000c/s example.
The shapes of all the spectrums for rectangular pulses in Fig. 10 are the same, only the scale has varied.

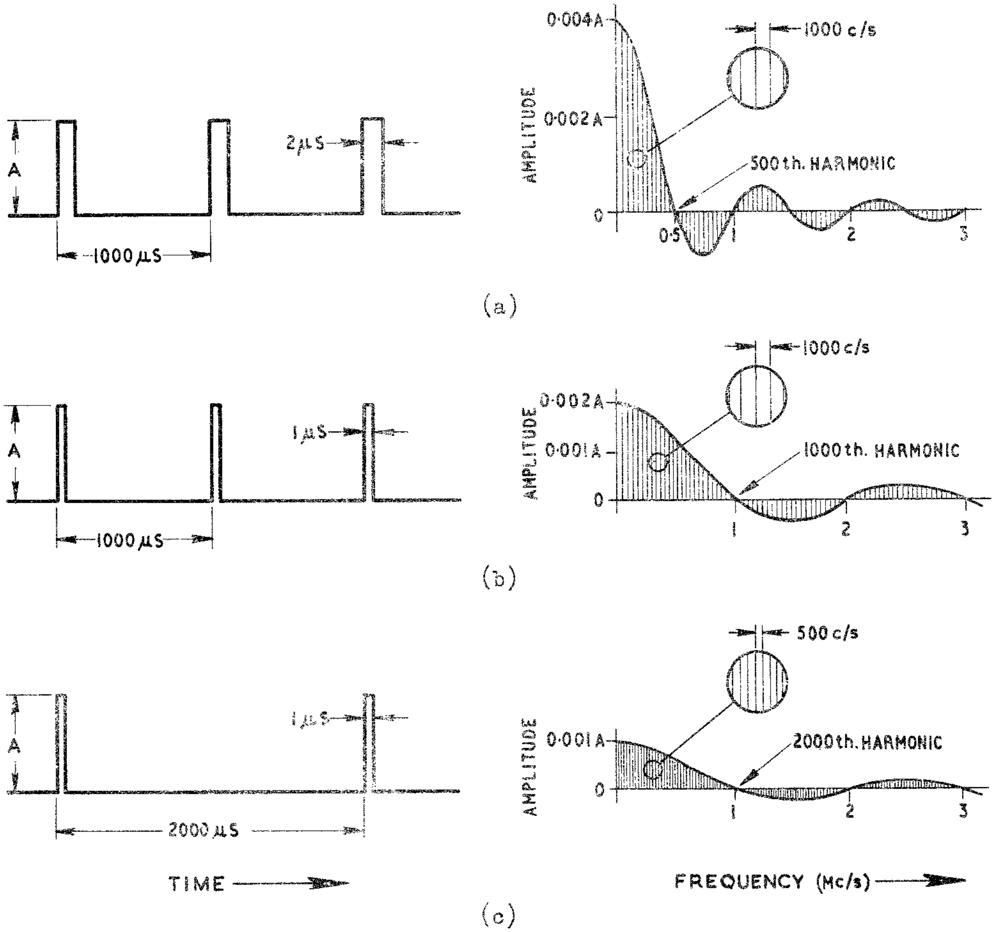
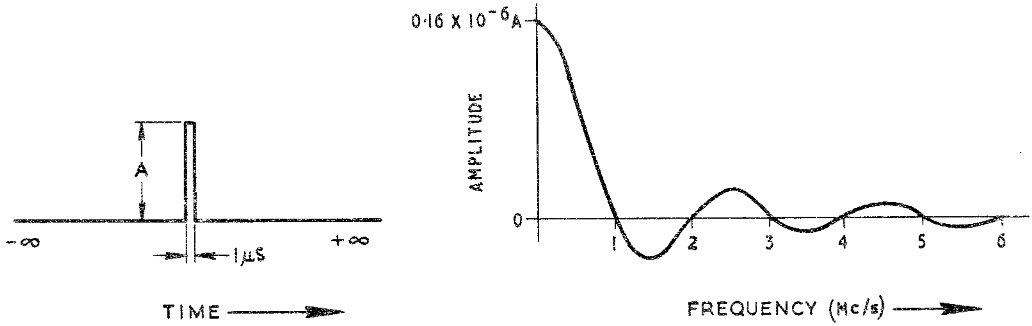



FIG. 10. FREQUENCY SPECTRUMS FOR SHORT DURATION PULSES.

Summarising, Fig. 10 shows that the components of the rectangular signal are spaced at frequency intervals equal to the repetition frequency of the pulses, and that the amplitude falls to zero at a frequency equal to $\frac{1}{\text{pulse duration}}$ and multiples of that frequency, independent of the pulse repetition frequency.

3.4 Transient Pulses. Consider that the repetition rate is decreased to the extreme. Only one pulse is produced and no other disturbances occur from an infinite time before ($-\infty$) to an infinite time after ($+\infty$) as shown in Fig. 11a. The shape of the spectrum is the same as before with the components falling to zero at a frequency equal to $\frac{1}{\text{pulse duration}}$, but as the repetition frequency is actually zero, there is no space between the components. That is, all frequency components enclosed by the "damped sine wave" shape are present (Fig. 11b). As can be expected, the amplitude of each component is reduced because there are so many components combining to produce the final amplitude.

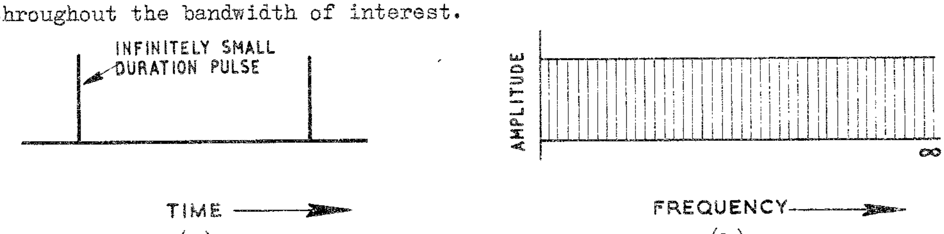


(a) (b)

FIG. 11. FREQUENCY SPECTRUM OF A TRANSIENT PULSE.

3.5 Infinitely Short Pulses. In a theoretical case of decreasing the length of the pulse so that its duration is infinitely small (Fig. 12a), the first zero in the frequency spectrum of the pulse is increased to infinity. In other words there is no zero, and the amplitude of the components is constant at all frequencies as shown in Fig. 12b.

This condition is approached in a practical case when the band of frequencies of interest has a high frequency limit that is much lower in frequency than the first zero of the pulse spectrum. When this occurs, component frequency amplitude are almost constant throughout the bandwidth of interest.



(a) (b)

FIG. 12. FREQUENCY SPECTRUM OF A PULSE OF ZERO DURATION.

3.6 Rise Time. In practice we are more interested in pulses with a finite rise time. It is necessary to know the bandwidth that is required to transmit a pulse with a specified rise time without degradation, and what is the fastest rise time that is possible when a pulse is passed through a system with a limited bandwidth.

Inspection of the frequency spectrum of the short duration rectangular pulses considered shows that the majority of the energy present in the components of the spectrum is at lower frequencies than the first zero. If only these frequencies are transmitted the pulse can still be identified as being present but its rise time is very poor.
To produce a pulse with a rise time that is small compared with the pulse duration, many higher frequency components are required.

Consider a simple "trapezoidal" waveform as in Fig. 13a. The amplitudes of the frequency components for this pulse train are dependent on both the half amplitude duration and the rise time of the pulse. There is a zero in the frequency spectrum at a frequency equal to $\frac{1}{\text{pulse h.a.d.}}$ and also at a frequency equal to $\frac{1}{\text{pulse rise time}}$. Normally, the rise time is much shorter than the pulse duration and so the first zero in the spectrum caused by the pulse duration is at a much lower frequency than the first zero caused by the pulse rise time.

To produce a pulse with a finite rise time, frequency components up to half the frequency of the first zero in the spectrum caused by the rise time must be included.

$$\therefore \text{ Highest frequency of a rectangular pulse} = \frac{1}{2 \times \text{rise time}} \quad \text{where} \quad \begin{matrix} \text{frequency} = \text{c/s.} \\ \text{rise time} = \text{secs.} \end{matrix}$$

Any system capable of transmitting a pulse with negligible change of rise time must pass frequencies to $\frac{1}{2 \times \text{rise time}}$.

The spectrum zero that is caused by the rise time is independent of any changes in the spectrum caused by the pulse duration, so that when the pulse duration is greater than the rise time, the frequency band required to pass the pulse with negligible degradation, depends only on the rise time. The same bandwidth is required for a short duration pulse as for a single step between two levels, if the rise time is the same in each case.

Although the pulse consists of many frequencies, in a simple consideration the highest frequency can be thought of as causing the shape of the pulse transition. In Fig. 13b, the total transition time is a half cycle of a sine wave at approximately the highest frequency passed by the system.

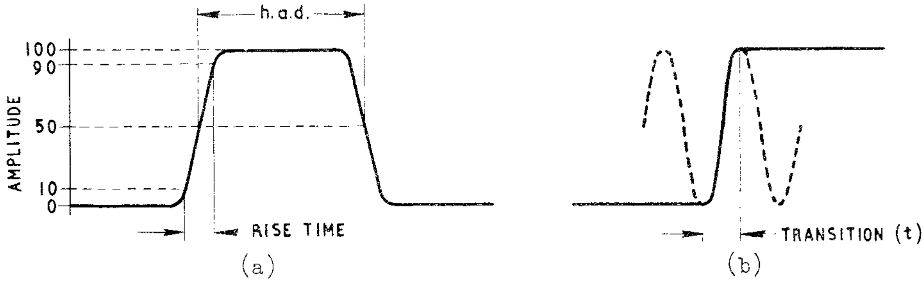



FIG. 13. PULSE OF FINITE RISE TIME.

If the time for this half cycle is t secs., the time for a full cycle is 2t secs., and the highest frequency that is responsible for the transition is $\frac{1}{2t}$ c/s. Considering the rise time as approximately t secs., the highest frequency required from the formula is $\frac{1}{2t}$ c/s. This coincides with the frequency representing the transition in the simple example. There are components present at higher frequencies, but for practical pulses of finite rise time, they are small in amplitude.

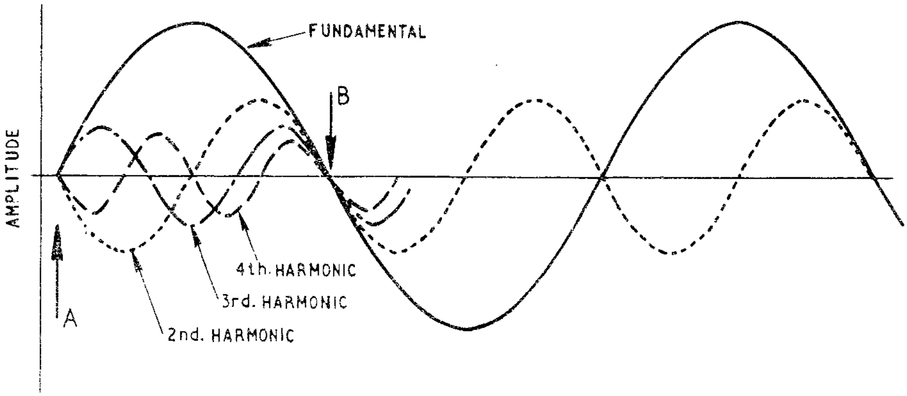
As an example, the highest frequency component of any importance that is present in a pulse with a 0.1µS rise time is -

$$\text{Highest frequency of pulse} = \frac{1}{2 \times \text{rise time}} = \frac{1}{2 \times 0.1}\text{Mc/s} = 5\text{Mc/s.}$$

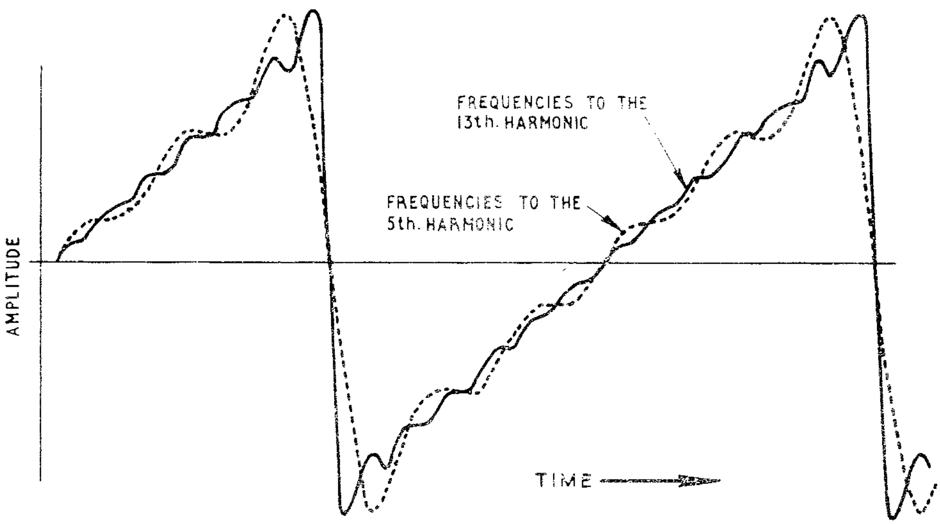
Relating this to the amplitude-frequency response of a circuit, if an ideal rectangular pulse is passed through a circuit and the output pulse rise time is 0.1µS, the bandwidth of the circuit is approximately 5Mc/s, and conversely, if the circuit bandwidth is 5Mc/s the output pulse rise time is 0.1µS (Rise Time = $\frac{1}{2 \times \text{Highest Frequency}}$). These statements are only approximate as the actual shape of the output waveform depends on the shape of the amplitude-frequency and phase-frequency responses, particularly near cut-off.

3.7 Sawtooth Waveforms. A sawtooth wave contains sine wave components including all harmonics of the fundamental repetition frequency, with phase relationships as indicated in Fig. 14a. Examining the phase relationship at time A, the fundamental and odd harmonics are positive going and the even harmonics are negative going. At time B, all components are negative going and the combination of the components at this time is responsible for the rapid retrace of the sawtooth. The amplitudes of the harmonics are the inverse of their order, i.e. the 2nd harmonic is half the amplitude of the fundamental and the 100th harmonic is 1% of the fundamental.

As more components are included in the resultant, the waveform produced approaches closer to an ideal sawtooth. Examples in Fig. 14b include components to the 5th and 13th harmonics only. Consequently, the linear forward trace of the sawtooth contains ringing, particularly at the ends, and the retrace time is not zero.




(a)




(b)

FIG. 14. COMPONENTS OF A SAWTOOTH WAVE.

The departure of the forward trace from being linear is of major importance and for this section of the wave to be satisfactory, harmonics to approximately the 50th must be included. Even under these conditions the retrace occupies a significant portion of one cycle of the wave, but this retrace time can be accepted and is of no great disadvantage. Actually, it is often a disadvantage to have a very fast retrace, as high peak voltages will be generated in inductive circuits.

## 4. D.C. COMPONENT.

4.1 A direct current component is present in a wave when more electrical energy is transferred in one direction than in the opposite direction. This means that the average of the algebraic sum of the instantaneous values of the wave is not zero. For some applications, the D.C. component of a waveform conveys important information. One particular case is for the correct reproduction of a television picture. The D.C. component of video signals is considered in Section 4 of the paper "Composite Video Signals".

The sum of the instantaneous values of a wave gives a result that is proportional to the area enclosed by the wave and the zero axis. With an A.C. signal, the average of the instantaneous values is zero. This is shown for sine, square, and sawtooth waves in Figs. 15a, b and c, where the area above zero equals the area below zero. In Figs. 15d, e and f, the same waveforms have a D.C. component added and this is verified by examination of the areas of the positive and negative sections of each wave. Fig. 15d has a positive D.C. component to the extent that never at any time is the wave negative, Fig. 15e has a resultant positive D.C. component, and Fig. 15f has a resultant negative D.C. component.

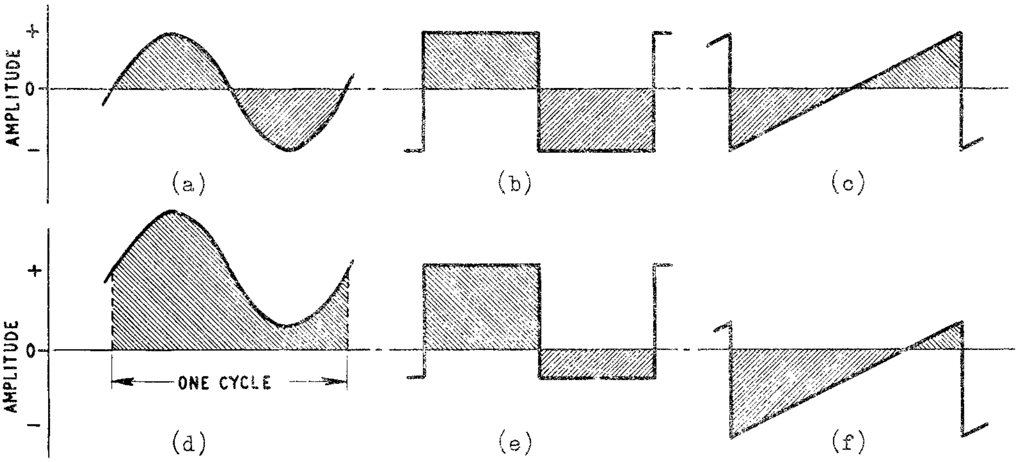



FIG. 15. WAVEFORMS WITH AND WITHOUT D.C. COMPONENT.

4.2 Magnitude of D.C. Component. The average of the instantaneous values of a wave is actually the magnitude of the D.C. component of the wave. Consider a positive rectangular pulse train with an amplitude of 10 Volts as in Fig. 16a. The wave has no negative section and the area of the positive section is 10 X 1t. This is the only section of the wave that has any magnitude for one complete cycle or a time of 8t, therefore dividing the area by 8t gives the average amplitude over a cycle. As the cycles are repetitive this is also the average for the complete wave.

$$\therefore \text{ The average amplitude } = \frac{10 \times 1t}{8t}$$

$$= 1.25 \text{ Volts } = \text{ The D.C. Component Amplitude.}$$

To prove that this is the average value of the wave, consider the areas of the wave above and below the 1.25 volt average line as in Fig. 16b.

The area above the average = 8.75 X 1t

= 8.75t.

The area below the average = 1.25 X 7t

= 8.75t = area above average.

The signal then can be divided into a D.C. component of 1.25V and a rectangular A.C. component with peaks of amplitude at -1.25V and +8.75V (Figs. 16c and d).

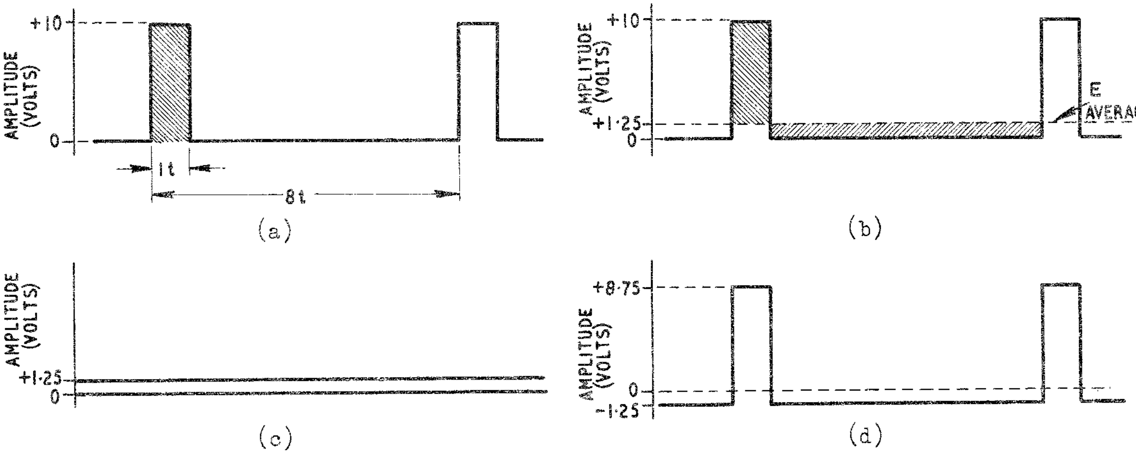



FIG. 16. WAVEFORM DIVIDED INTO COMPONENTS.

A general formula for calculating the D.C. component or average value of a signal is -

$$\text{Average Amplitude} = \frac{\text{Positive Area} - \text{Negative Area}}{\text{Total Time}}$$

This formula is used in the following examples and applies independent of whether the resultant component is positive or negative. In example (2) where the signal is always negative, the complete area is calculated by considering sections of the total area that have convenient shapes.

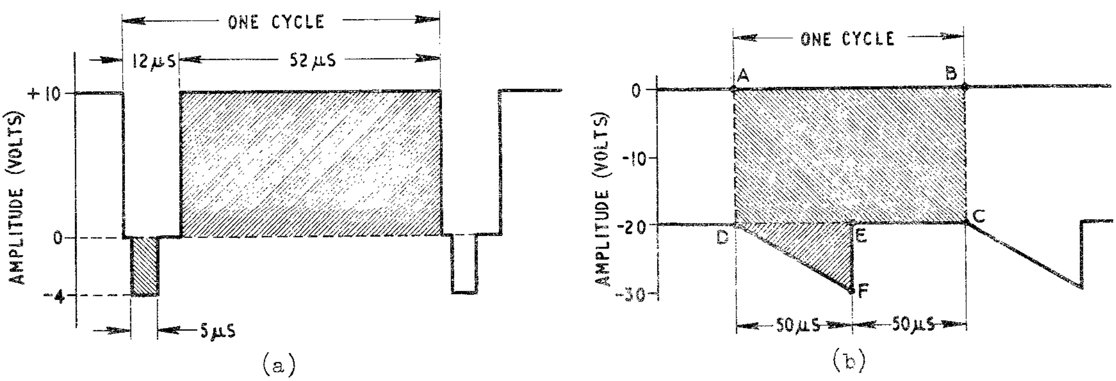



FIG. 17. WAVEFORMS WITH D.C. COMPONENT.

Example 1. What is the value of D.C. component for the signal in Fig. 17a?

Positive Area of one cycle = 10 X 52 units

= 520 units.

Negative Area of one cycle = 4 X 5 units

= 20 units.

Average Voltage (D.C. Component) = $\frac{520 - 20}{64}$

= +7.8 Volts

Answer:- D.C. Component of Fig. 17a = +7.8 Volts.

Example 2. Calculate the average value of the waveform in Fig. 17b.

Positive Area = 0

Negative Area = Area ABCD + Area DEF

= $(20 \times 100) + (\frac{10 \times 50}{2})$ units

= 2000 + 250 = 2250 units.

Average Voltage = $\frac{\text{Positive Area} - \text{Negative Area}}{\text{Total Time}}$

= $\frac{0 - 2250}{100}$

= -22.5 Volts

Answer:- Average Value of Fig. 17b = -22.5 Volts.

## 5. RESISTOR-CAPACITOR CIRCUITS.

5.1 Voltage Variation with Time. In Section 3, the frequency components of pulses were examined. The main use for this information is in considering the bandwidth required to transmit pulses. Examination of pulse circuits by considering the frequency components and the responses of associated circuits is difficult and usually ends up as a mathematical treatment which is not useful as a simple explanation.

Pulse techniques generally rely on the time for charge and discharge of capacitors and inductors associated with resistors. The majority of pulse circuits, then, are more easily explained by considering each pulse as a change in the D.C. voltage applied to the circuit, and by considering the charge or discharge that occurs because of the voltage change. The voltage or current in the circuit is therefore considered on a time scale.

In this paper, and in other papers where a distinction is required, capital letters are used as symbols to designate fixed values such as the supply voltage ($E_S$). Lower case letters are used as symbols for quantities that are varying with time; for example, the instantaneous voltage across a capacitor is designated $e_C$.

5.2 Charge of Capacitor. When a source of e.m.f. is connected to a circuit consisting of a resistor and an uncharged capacitor in series (Fig. 18a), the capacitor does not charge instantly to the supply voltage ($E_S$). At the instant of connection, the voltage across the capacitor is zero as it has no charge, and the voltage across the resistor is equal to the supply voltage. The current through a resistor is always equal to the P.D. across it, divided by its resistance, therefore the initial current in the circuit is equal to $\frac{E_S}{R}$. As current flows, a charge is built up in the capacitor, the quantity of charge (Q) being dependent on the current and the time ($Q = It$).

5.3 Exponential Charge Curves. In the circuit considered, the charging current cannot continue at its initial rate. As the current charges the capacitor, a voltage is developed across the capacitor, and the voltage across the resistor decreases to the same extent. The resistor voltage at any instant ($e_R$) is the difference between the supply voltage and the capacitor voltage at that instant ($e_C$); ($e_R = E_S - e_C$). The charging current therefore also decreases and the charging current at any instant is $i = \frac{E_S - e_C}{R}$. The decrease in the charging current causes a slower rate of increase of voltage on the capacitor.

The change of the values of current and voltage in the circuit is shown in Fig. 18b. These curves have a shape that is described as exponential. The current (i) is at its maximum value ($I_M$) and equal to $\frac{E_S}{R}$ at the instant the circuit is completed, and as the capacitor charges, the current gradually decays towards zero. The voltage across the resistor is proportional to current and is equal to the supply voltage when the circuit is first completed, and then decays towards zero. The voltage across the capacitor is initially low and rises exponentially to approach the supply voltage.

The charge current in Fig. 18a is anticlockwise in the circuit, and the voltages across the capacitor and resistor have the polarities as shown.

5.4 Time Constant. Independent of the values of the circuit components, the curves of voltage and current against time have the same shape. However, the actual charging time depends on the values of the components, and the scales must be arranged to suit the component values and the supply voltage. When the resistance is increased, the maximum current is reduced, and since $Q = It$, a longer time is required to achieve the same charge. Because $Q = CE$, increasing the value of capacitance increases the Q for a given supply voltage, and therefore also increases the charging time (for any given value of series resistance).

The product of capacitance and resistance (CR) is known as the time constant of the circuit (Symbol - $\tau$ - Tau).

In a resistor-capacitor circuit - $\tau = CR$ where $\tau$ = time constant in seconds, C = capacitance in farads, R = resistance in ohms.

An important feature of the exponential curves in Fig. 18 is that after a time equal to the circuit time constant, $e_C$ has increased to approximately 63.2% of the applied voltage change, and $e_R$ and i have decreased by approximately 63.2% to approximately 36.8% of their respective maximum possible changes.

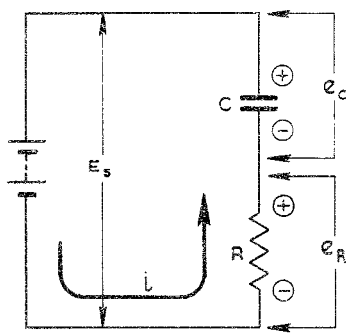
This leads to a practical definition of time constant:-

> The time constant of a resistor-capacitor (R-C) circuit is the time for the capacitor voltage to change by 63.2% of the applied voltage change.

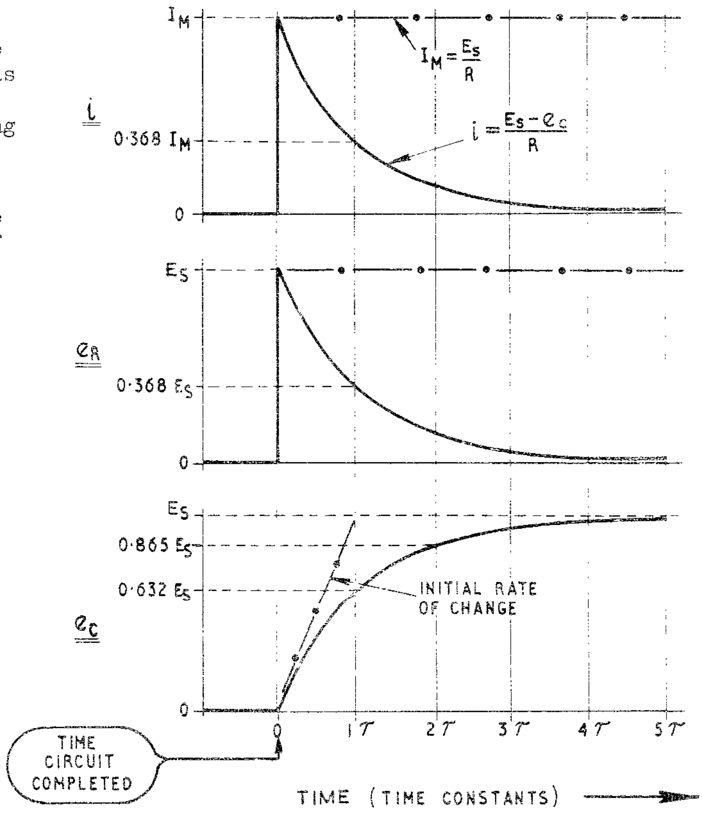
In theory, a capacitor never becomes fully charged, since for equal time intervals the voltage always changes by the same percentage of the remaining voltage which determines the current in the circuit, that is the voltage across the resistor. As an example, if 100 volts is applied to a resistor-capacitor circuit, after a time equal to one time constant $e_C$ = 63.2V and $e_R$ = 36.8V.

After a further time again equal to one time constant, the capacitor voltage has increased by a further 63.2% of the remaining 36.8 volts that is determining the charging current, i.e. 23.26V, and is approximately 86.46V.

After five time constants, however, the capacitor voltage change exceeds 99% of the applied voltage change and is considered for practical purposes as being completely charged. Values for other charge times are considered in Sections 7 and 8.




(a)




(b)

FIG. 18. CHARGE OF A CAPACITOR THROUGH A RESISTOR.

As a matter of interest, if the initial current could continue, the capacitor voltage would continue to increase at its initial rate and would be charged to the supply voltage in a time equal to the circuit time constant. This relationship can be proved as follows:-

$$\text{Initial current } (I_M) = \frac{E_S}{R}.$$

When capacitor is charged to $E_S$, $Q = CE_S$.

Because $Q = It$, the time (t) to charge the capacitor at the initial rate

$$= \frac{Q}{I}$$

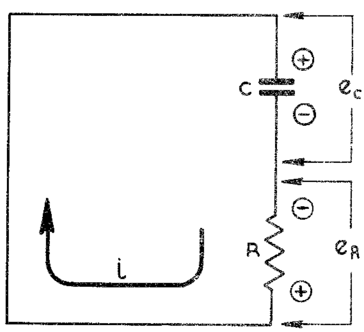
$$= \frac{CE_S R}{E_S} = CR = \tau$$

5.5 Discharge of Capacitor. Consider that the capacitor in the circuit of Fig. 18a is fully charged to the supply voltage ($E_S$) and that the source of e.m.f. is removed and replaced by a short circuit as in Fig. 19a. The capacitor voltage is directly across the resistor and causes a current in the circuit in a clockwise direction. This is in the opposite direction to the initial charging current, and therefore the voltage across the resistor is also opposite in polarity to the voltage during charging.

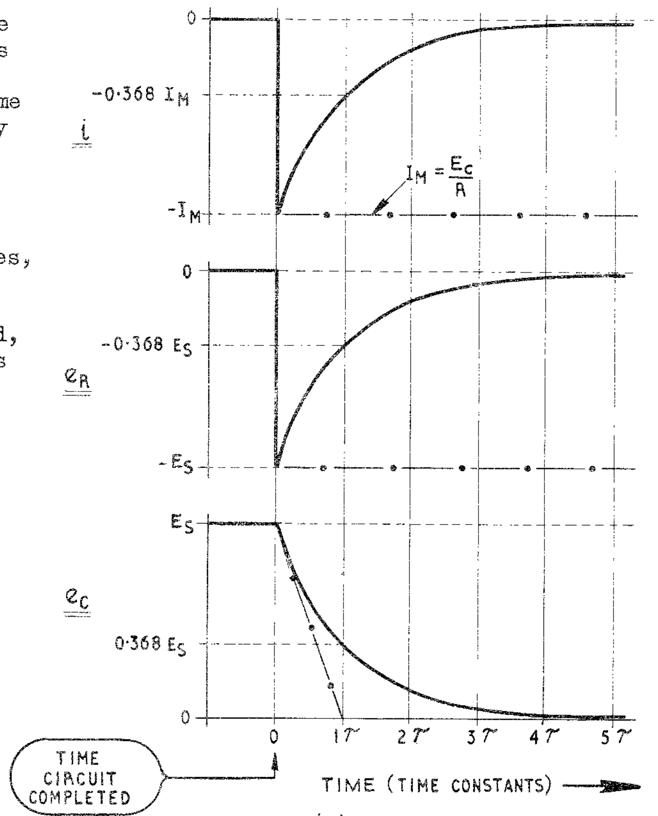
The magnitude of current in the circuit at the instant of application of the short is equal to $\frac{E_S}{R}$, which is the same as the instantaneous peak of current when the charging voltage was first applied. As current flows the capacitor discharges and the voltage across the capacitor, and therefore also across the resistor, reduces. This reduces the value of discharge current, which reduces the rate of fall of the capacitor voltage.

Both the voltage and current in the circuit decay exponentially towards zero as shown by the curves in Fig. 19b. As before, after one time constant the values have changed by 63.2% to 36.8% of their respective maximum values.

After five time constants the capacitor is, for practical purposes, fully discharged and the discharge current is zero. Energy is stored in the capacitor when it is charged, and during discharge this energy is dissipated in the resistor.




(a)

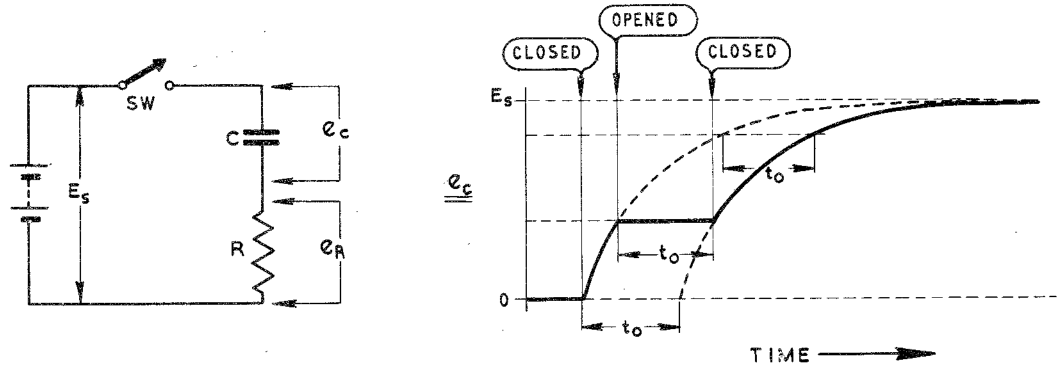



(b)

FIG. 19. DISCHARGE OF A CAPACITOR THROUGH A RESISTOR.

5.6 Residual Charge. So far we have considered the charge of a capacitor from zero to a maximum value, and then the discharge back to zero. In many practical applications, neither of the two extremes of voltage on the capacitor are zero.

Consider that the switch in Fig. 20a is closed for a short period, and then opened and closed again. When the circuit is first completed the capacitor commences to charge. At the opening of the switch no discharge circuit is provided and the capacitor voltage is maintained. On the closing of the switch again, the capacitor charge is again increased, and follows exactly the same values that it would have, if the circuit had not been broken (Fig. 20b). The only difference is the displacement in time of the section of the curve by an amount equal to the time of the break. The rate of increase of charge is dependent on the voltage acting in the circuit at any time, and this is a characteristic of the exponential shape; no matter where the charge recommences on the curve, the remaining section always has the same shape.




(a) (b)

FIG. 20. INTERRUPTION OF CAPACITOR CHARGE.

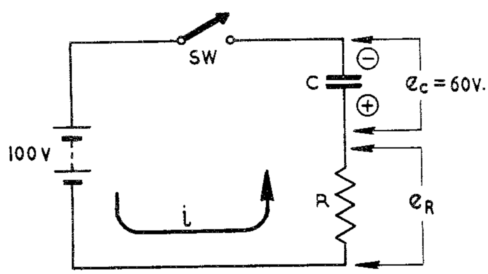
This applies even when the change of voltage on the capacitor undergoes a change of polarity. In Fig. 21a, the capacitor is initially charged to 60 volts. When the switch is closed current flows to discharge the capacitor, and then charge it to 100 volts in the opposite direction. The complete discharge-charge curve is part of one exponential curve and there is no disjointed appearance as the curve passes through the point of zero voltage (no charge) on the capacitor.

At the instant the circuit is completed, the current is determined by the algebraic sum of the sources of e.m.f. present in the circuit. In this case it is 160 volts, as the two voltages present are aiding.

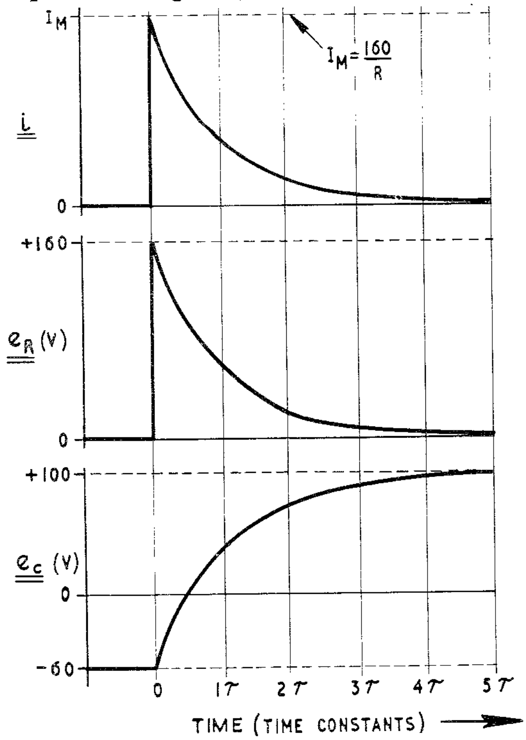
The change of current and voltages in the circuit is shown in Fig. 21b. If the capacitor's initial voltage before the switch is closed (-60 volts) is taken to be the reference voltage, then when the circuit is closed the capacitor charges from this reference to the maximum 160 volts more positive, and the curve is the same shape as in Fig. 18, for the simple example with no initial charge.

The current and the resistor voltage changes are also similar to the simple example, with maximum amplitudes as if there was no initial charge on the capacitor, and the battery voltage was 160 volts.

In calculations relating to charge and discharge, the change of voltage that is possible in the circuit, that is, the resultant voltage causing the charging current at the time of reference, is more important than the supply voltage of the circuit.




(a)




(b)

FIG. 21. RESIDUAL CHARGE ON CAPACITOR.

## 6. INDUCTOR-RESISTOR CIRCUITS.

6.1 "Charge" of Inductor. When a source of e.m.f. is connected to a circuit consisting of an inductor and resistor in series as in Fig. 22a, the current does not rise instantly to the maximum value. At the instant the circuit is completed there is no current, and therefore the voltage across the resistor ($e_R$) is zero. The current attempts to change and a self induced e.m.f. is developed across the inductor which opposes the applied e.m.f., and limits the initial rate of change of current. To preserve the law that the algebraic sum of the voltages in the circuit must be zero ($E_S = e_L + e_R$), as $e_R$ is zero, the voltage across the inductor ($e_L$) equals $E_S$. The rate of change of current at this instant must then be such as to produce an induced voltage equal to the applied voltage.

6.2 Exponential Charge Curves. The increase in current through the inductor cannot continue at its initial rate in this circuit. When current commences, a voltage drop is produced across the series resistor. This means that the voltage across the inductance must decrease ($e_L = E_S - e_R$). The decrease in $e_L$ is because of a reduced rate of change of current, and this causes a reduced rate of rise of voltage across the resistor.

The curves of voltage and current with time are exponential and are shown in Fig. 22b. The circuit current which is initially zero, increases exponentially towards a maximum value. When it has reached its maximum value, there is no change of current, and therefore no voltage across the inductor. The total supply voltage is then across the resistor and the maximum value of current ($I_M$) is equal to $\frac{E_S}{R}$. The resistor voltage ($e_R$) is proportional to the current, and increases from zero to approach $E_S$. The voltage across the inductor ($e_L$) steps from zero to equal $E_S$ when the circuit is first completed, and then decays exponentially towards zero.

Notice that the curve for voltage across the resistor in an L-R circuit is the same as the voltage across the capacitor of an R-C circuit, and the voltage across the inductor in an L-R circuit varies in the same way as the voltage across the resistor in an R-C circuit.

6.3 Time Constant. Again, as with the R-C case, the shapes of the curves of voltage and current are independent of the component values, but the actual time for the current in the circuit to rise is determined by the component values, so that suitable scales must be included.

The induced e.m.f. developed across an inductor ($e_L$), is dependent on the rate of change of current and the inductance (L).

$$e_L = \text{rate of change of current} \times L$$
$$= \frac{di}{dt}L.$$

(This is the basis of the definition of the henry, the unit of inductance. An induced voltage of one volt is produced by current changing at the rate of one amp per second in a coil with an inductance of one henry.)

When the circuit is first completed, $e_L = E_S$. Therefore, if the value of inductance is increased, the initial rate of change of current must decrease to maintain $e_L = E_S$ and it takes longer for the current to build up. When the resistance of the circuit is increased, the maximum current possible in the circuit is decreased ($I_M = \frac{E_S}{R}$), and for a given value of inductance and supply voltage, it takes a shorter time for the current to approach its maximum value.

The ratio of inductance to resistance ($\frac{L}{R}$) is known as the time constant ($\tau$) for inductor-resistor (L-R) circuits.

In an inductor-resistor circuit - $$\tau = \frac{L}{R}$$ where

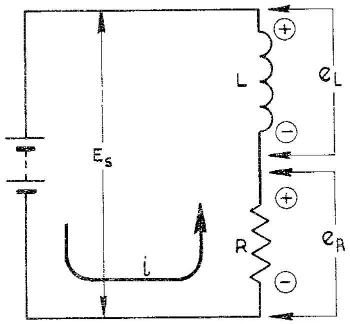
$\tau$ = time constant in seconds
L = inductance in henries
R = resistance in ohms.

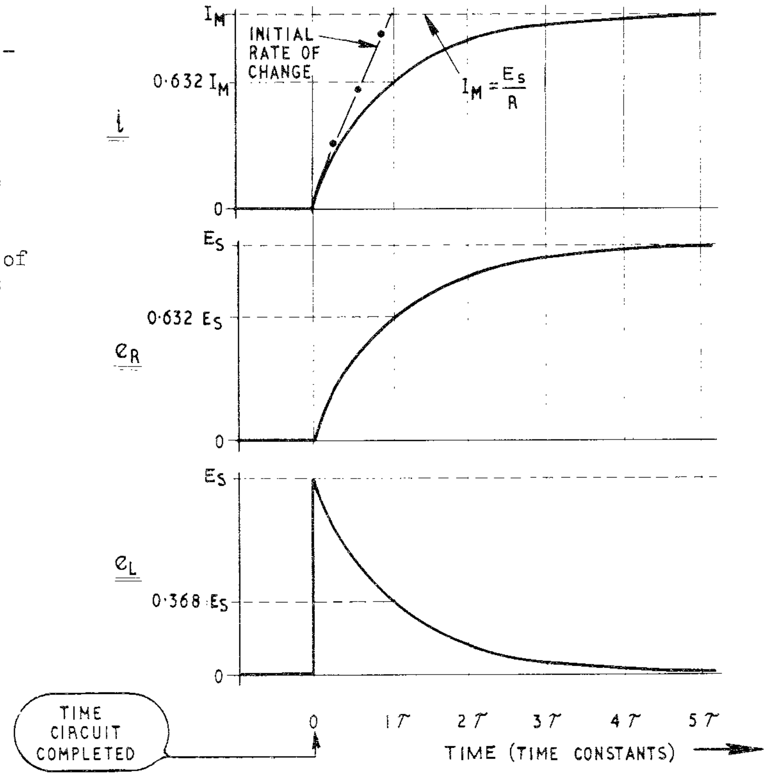
In Fig. 22b, after one time constant, $e_L$ has reduced by 63.2% to 36.8% of its maximum value, and $e_R$ and the current in the circuit (i) have increased to 63.2% of their respective maximums.

The practical definition of time constant for L-R circuits then is -

> The time constant of a circuit containing inductance and resistance is the time for the current in the circuit to change by 63.2% of its maximum change.

As with R-C circuits, the changes of voltage and current in the circuit are considered as being completed after five time constants.




(a) (b)

FIG. 22. "CHARGE" OF AN INDUCTOR THROUGH A RESISTOR.

For interest, if the initial rate of increase of current could continue, it would reach the maximum value in a time equal to the time constant. This relationship can be proved as follows:-

$$e_L = \frac{di}{dt}L$$

$$\therefore \frac{di}{dt} = \frac{e_L}{L}$$

$$\text{Initial voltage across inductor} = E_S$$

$$\therefore \text{Initial rate of change of current} = \frac{E_S}{L}$$

$$I_M = \frac{E_S}{R}$$

$$\therefore \text{Time (t) to reach the maximum value } I_M = \frac{I_M}{\text{rate of change of current}}$$

$$= \frac{E_S L}{R E_S}$$

$$= \frac{L}{R} = \tau$$

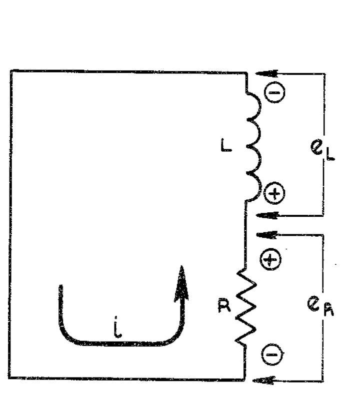
6.4 "Discharge" of Inductor. When a current is passing through an inductor, energy is stored in the magnetic field produced. Consider that the source of e.m.f. is removed and replaced by a short circuit without breaking the circuit (Fig. 23a). The inductance opposes any change of current in the circuit and if no losses were present the current would continue. With resistance in the circuit, energy stored in the magnetic field is dissipated in the resistance, and the magnetic field and current producing it gradually fall to zero.

At the instant the discharge circuit is completed, the current is still $I_M$, so that the voltage across the resistor is still equal to $E_S$. This voltage is the induced voltage produced by the current attempting to change in the inductor, but as the inductor is now the source of e.m.f., the voltage is of opposite polarity to the voltage during the build up of the magnetic field as shown in Fig. 23a, and therefore $e_L = -E_S$. To produce a voltage across the inductor with the required polarity and magnitude, the current in the circuit must be decreasing at the same rate as it increased during charging, that is, if the initial rate of change is continued the current will be zero in one time constant.

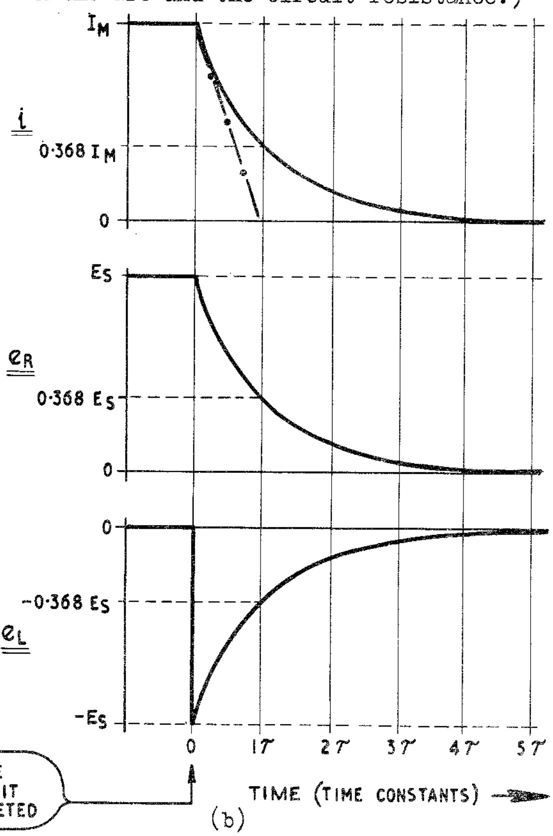
However, the initial rate of change cannot continue, for when the current decreases there is less voltage drop across the resistor, and therefore less induced voltage across the inductor. This reduction of inductor voltage is caused by a reduced rate of change of current. Fig. 23b shows the discharge curves, which are again exponential, decreasing from their maximum values to 36.8% in one time constant, and reaching a practical zero in five time constants.

(When an inductive circuit is opened, and not first shorted, the series resistance in the circuit becomes infinite, and therefore the time constant $(\frac{L}{R})$ of the circuit during the collapse of the magnetic field, is zero. That is, the initial rate of change is infinite and so the induced e.m.f. is infinite. In practice, the voltage increases instantly to the extent that it is able to arc across the opening contacts and the energy stored in the circuit is dissipated in the arc and the circuit resistance.)

6.5 Initial Current. We have considered the change of current in an inductance from zero to a maximum and back to zero. In the inductive circuit there may be some initial current and any variation of this current through a resistance follows the exponential shape in the same way as examined for voltage in a capacitive circuit. In these cases we are interested in the change of current rather than the actual value of current.




(a)




(b)

FIG. 23. DISCHARGE OF AN INDUCTOR THROUGH A RESISTOR.

## 7. UNIVERSAL TIME CONSTANT CHART.

7.1 An observation of the curves for charge and discharge of capacitors and inductors shows that all changes of current and voltage in the circuits have an exponential shape, and follow two basic curves -

(i) The magnitude of voltage or current decreases from either a positive or a negative maximum value, quickly at first, and then at an increasingly slower rate as it approaches zero or the reference value.

(ii) The magnitude of voltage or current increases quickly at first and then at an increasingly slower rate as it approaches a maximum value which is either positive or negative.

A universal time constant chart is drawn in Fig. 24, which includes the two curves required. The horizontal scale of time is related to the time constant of the circuit, and calculations are made by relating actual time to the circuit time constant as calculated from the formulae $\tau = CR$ or $\tau = \frac{L}{R}$. The vertical scale is calibrated so that at any time the voltage or current is indicated as a fraction of the maximum voltage or current. The actual value of voltage or current at the instant required is then found by multiplying the maximum value by the factor determined from the correct curve on the graph, for the time required. With the chart, calculations with accuracies to two significant figures are possible, and examples are worked in paras. 7.2 and 7.3 to indicate how the chart is used.

When considering charge and discharge, we found that the sum of the voltage drops across the components at any instant is equal to the applied voltage. Here also the sum of the instantaneous values of curve "A" and curve "B" equals the maximum value of unity.

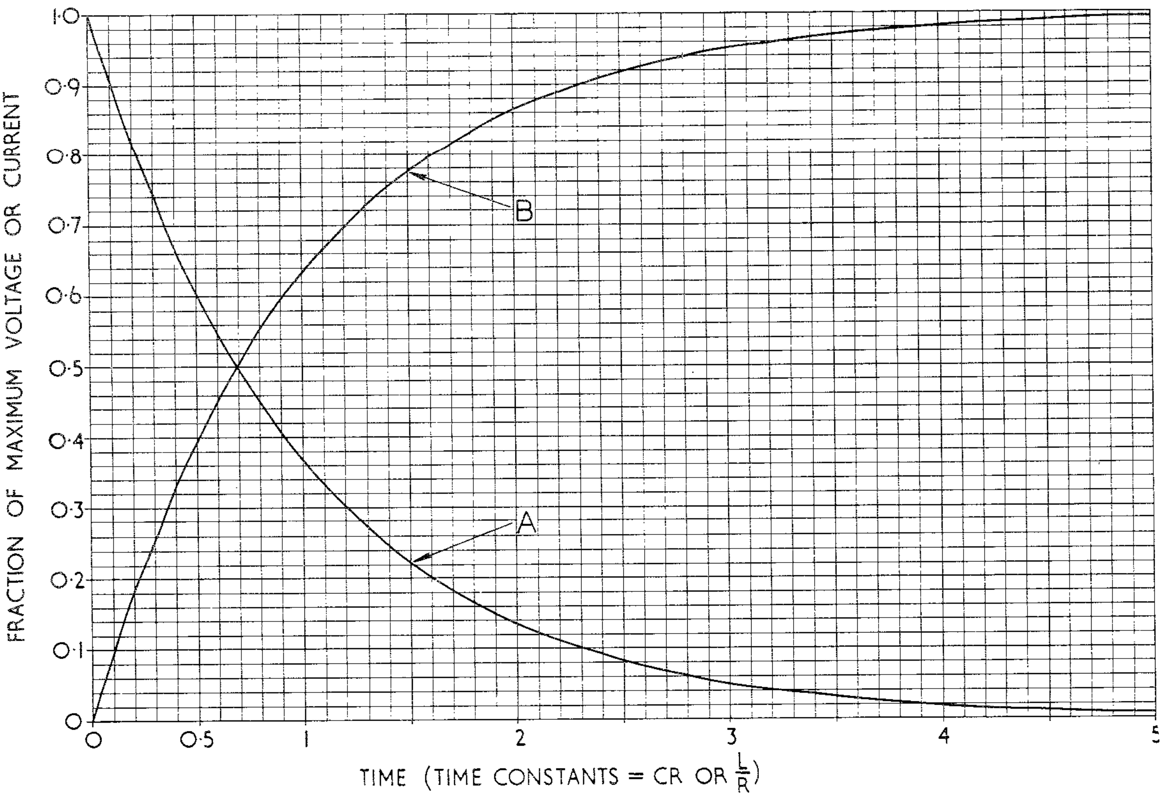
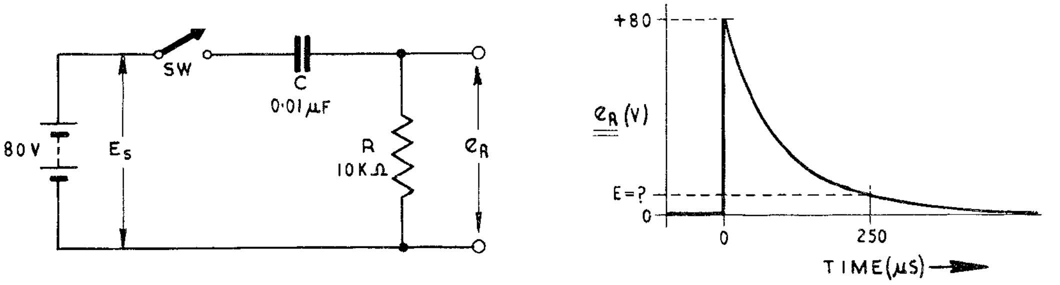



FIG. 24. UNIVERSAL TIME CONSTANT CHART.

7.2 The following examples set out typical ways of solving problems associated with R-C and L-R circuits, and make use of the universal time constant chart.

Example 3. In the circuit of Fig. 25a, find the value of voltage across the resistor after the switch has been closed for 250µS.




(a) (b)

FIG. 25. R-C CIRCUIT FOR EXAMPLE 3.

(i) Estimate the shape of the required curve. When the circuit is completed the resistor voltage immediately steps to 80V and then decreases exponentially towards zero as in Fig. 25b. This shape corresponds to curve A of the universal time constant chart.

(ii) Calculate the time constant of the circuit -

$$\tau = CR$$

$$\tau = 0.01 \times 10^{-6} \times 10^{4} \times 10^{6}\ \mu S$$

$$= 100\mu S.$$

(iii) Express 250µS as a time (x) relative to the circuit time constant -

$$x = \frac{t}{\tau}$$

$$= \frac{250}{100} = 2.5 \text{ time constants.}$$

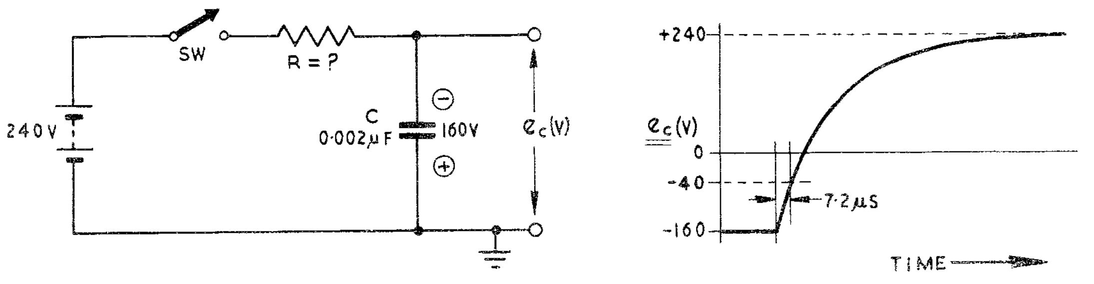
(iv) Read off the factor on the vertical scale corresponding to 2.5 time constants and curve A, i.e. approximately 0.08.

(v) Calculate the voltage -

$$e_R = 0.08 \times 80 = 6.4V$$

Answer:- Voltage across resistor after 250µS = 6.4V.

Example 4. A capacitor is charged so that the output voltage is -160V with respect to earth as in Fig. 26a. When the switch is closed the output voltage reaches -40 volts in 7.2µS. What is the value of the resistance in the circuit?




(a) (b)

FIG. 26. R-C CIRCUIT FOR EXAMPLE 4.

The capacitor charge is changing from a reference voltage of -160 volts to +240 volts as in Fig. 26b. At the instant the circuit is completed, the e.m.f. acting in the circuit, and therefore the maximum change of voltage possible is -

$$E_A = 160 + 240 = 400V.$$

The change from the reference to -40V = +120V.

This change expressed as a fraction of the maximum change $= \frac{120}{400} = 0.3.$

From curve B of the universal time constant chart, the change takes place in a time of approximately 0.36 time constants.

$$\therefore \; 7.2\mu S = 0.36 \text{ time constants}$$

$$\therefore \; \text{The circuit time constant} = \frac{7.2}{0.36} = 20\mu S.$$

$$\therefore \; R = \frac{\tau}{C}$$

$$= \frac{20 \times 10^6}{10^6 \times 0.002}$$

$$= 10,000\,\Omega$$

Answer:- Circuit resistance = 10,000Ω.

Example 5. A relay with a resistance of 1000Ω and an inductance of 5 henries will operate when 20mA of current flows through the coil. How long will it take for the current to increase to the operate value after the relay is connected to a 50 volt supply?

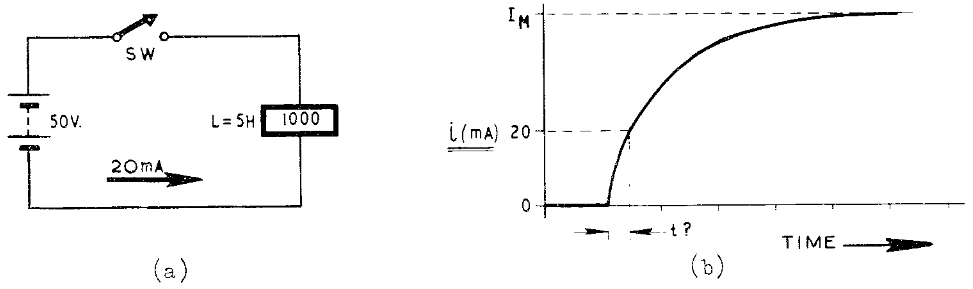



FIG. 27. L-R CIRCUIT FOR EXAMPLE 5.

The current rises exponentially to a maximum with a shape as in Fig. 27b.

$$I_M = \frac{E_S}{R}$$

$$= \frac{50 \times 10^3}{10^3} = 50mA.$$

$$\text{Fraction of } I_M \text{ required to cause operation} = \frac{20}{50} = 0.4.$$

From curve B of the universal time constant chart -

Time to reach 0.4 of Max. = 0.51 time constant.

$$\tau = \frac{L}{R}$$

$$= \frac{5 \times 10^3}{10^3} = 5mS.$$

$$\therefore \; \text{Time for current to reach 20mA} = 0.51 \times 5 = 2.55mS.$$

Answer:- Time for current to reach operate value = 2.55mS.

7.3 The following example illustrates a more involved calculation of a type likely to be encountered in practice.

Example 6. The switch in Fig. 28 is moved to position 2 for 10mS and then returned to position 1 for 10mS. The switching cycle is then continued at this regular rate. The time for each change-over is negligible.

Calculate the voltages present immediately before and after each switch operation, and draw graphs for the first two switching cycles (40mS) showing -

(i) Input voltage to the network ($e_{in}$) against time, and

(ii) Output voltage from the network ($e_{out}$) against time.

Indicate the calculated voltages on the graphs.

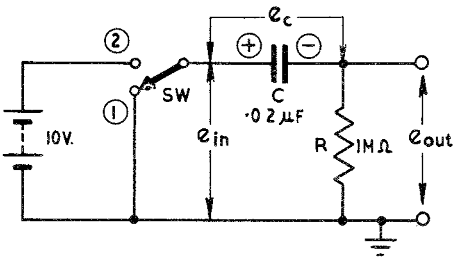



FIG. 28. CIRCUIT FOR EXAMPLE 6.

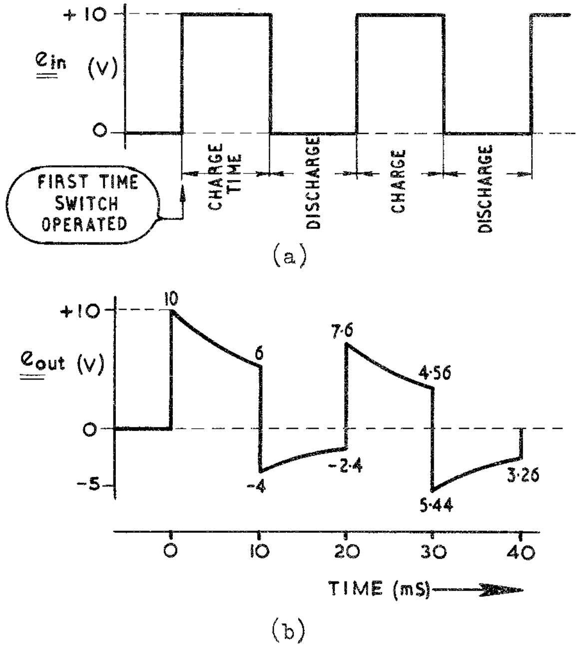



FIG. 29. ANSWER TO EXAMPLE 6.

(i) When the switch is in position 1, $e_{in}$ is zero and in position 2, $e_{in}$ is 10V. After the first operation 10mS is spent in each position and the graph of $e_{in}$ is as in Fig. 29a.

(ii) The time constant for both charge and discharge is 20mS ($\tau$ = CR). When the switch is in position 2, the capacitor charges for 10mS and when in position 1, the capacitor discharges for 10mS. Therefore, the charging time and the discharging time per cycle is each 0.5 time constants. Therefore in each 10mS, from curve A of the universal time constant chart, $e_{out}$ falls to approximately 0.6 of the voltage present at the start of this time.

When switch is first operated to position 2, there is no charge and therefore no voltage across the capacitor.

$\therefore$ $e_{out}$ = +10V.

After 10mS, $e_{out}$ = 0.6 X 10

= +6V.

$\therefore$ $e_C$ = 4V. (Polarity as in Fig. 28)

Switch changes back to position 1.

$\therefore$ $e_{out}$ = $e_C$

= -4V.

After 20mS, $e_{out}$ = 0.6 X -4

= -2.4V.

Switch changes to position 2 again.

The algebraic sum of the e.m.f's. in the circuit

= 10 - 2.4

= +7.6V.

= $e_{out}$

After 30mS, $e_{out}$ = 0.6 X 7.6

= 4.56V.

$\therefore$ $e_C$ = 5.44V.

Switch changes to position 1 again.

$\therefore$ $e_{out}$ = -5.44

After 40mS, $e_{out}$ = 0.6 X -5.44

= -3.264V.

The output voltage ($e_{out}$) is as in Fig. 29b and is shown on a common time scale with $e_{in}$.

## 8. EXPONENTIAL FUNCTIONS.

8.1 For more accurate calculations of the state of charge of a capacitor or inductor, the factor derived from the universal time constant chart can be obtained from tables of exponential functions. Even when the additional accuracy is not required, it is often more convenient to read a value from tables, than to estimate it from a graph by projecting the point on the graph to the two scales.

8.2 Exponential Formulae. The description of the charge and discharge curves as "exponential" suggests that their shapes are dependent on the exponent of a number, that is the index or the power to which a number is taken. The formulae describing the shape of the curves of the universal time constant chart are related to powers of the base of Napierian or natural logarithms ($\epsilon$).

Curve A in Fig. 24 is obtained by plotting the results obtained by substituting values for x in the formula.

$$y_A = \epsilon^{-x}$$

$$\text{that is} \quad y_A = \frac{1}{\epsilon^x}$$

In our application, y becomes the fraction of the maximum amplitude or the vertical scale, and x is the time expressed in time constants or the horizontal scale of the resulting curve.

When we substitute x = 1, the result gives the fraction of the maximum after one time constant.

$$y_A = \epsilon^{-1} \text{ or } \frac{1}{\epsilon^1}$$

$$= \frac{1}{\epsilon}$$

$$\text{The value of } \epsilon = 2.71828.$$

$$\therefore \ y_A = \frac{1}{2.71828}$$

$$= 0.3679.$$

This agrees with the value obtained from curve A after one time constant.

To find the value after two time constants we substitute x = 2.

$$\therefore \ y_A = \epsilon^{-2} \text{ or } \frac{1}{\epsilon^2}$$

$$= \frac{1}{2.71828^2}$$

$$= \frac{1}{7.3891}$$

$$= 0.1353.$$

Checking this against Fig. 24 shows that the curves agree to the accuracy possible.

The sum of the instantaneous values of curves A and B in Fig. 24, is equal to the maximum value of unity on the scale for the graph, so that the formula for curve B is obtained by subtracting the formula for curve A from 1.

$$\text{i.e.} \quad y_B = 1 - \epsilon^{-x}$$

$$\text{or} \quad y_B = 1 - \frac{1}{\epsilon^x}$$

$\therefore$ when x = 2, from the previous example -

$$\epsilon^{-2} = 0.1353$$

$$\therefore \ y_B = 1 - 0.1353 = 0.8647.$$

This again agrees with the universal time constant chart.

8.3 Table of Exponential Functions. When making calculations it is likely that the time of interest has no simple relationship to the time constant. It may be, for example, a time equal to 2.4 time constants. This makes the value of $\epsilon^{-2.4}$ harder to calculate. For convenience, tables are available which normally include values for $\epsilon^{x}$ and $\epsilon^{-x}$ for various values of x to approximately 6. Some tables of exponential functions include in addition to the above, a column with values of $1 - \epsilon^{-x}$. This makes it more convenient to find the values on curves that are increasing exponentially towards a maximum value.

An example of a section of a table is shown in Table 2. To find the multiplying factor to determine a value after 2.4 time constants from the initiation of the change for an exponential curve which is approaching zero, we locate x = 2.4 and read off $\epsilon^{-x} = 0.0907$.

To find the time for a curve to increase exponentially from zero to, for example, 20% of the maximum value, we look for 0.2 in the $1 - \epsilon^{-x}$ column. This gives a value for x between 0.22 and 0.23 that is approximately 0.223. Therefore it takes 0.223 time constants for the exponential curve to reach 20% of the maximum value.

| x | $\epsilon^{x}$ | $\epsilon^{-x}$ | $1-\epsilon^{-x}$ | x | $\epsilon^{x}$ | $\epsilon^{-x}$ | $1-\epsilon^{-x}$ | x | $\epsilon^{x}$ | $\epsilon^{-x}$ | $1-\epsilon^{-x}$ | x | $\epsilon^{x}$ | $\epsilon^{-x}$ | $1-\epsilon^{-x}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| .00 | 1.0000 | 1.0000 | .0000 | .50 | 1.6487 | .6065 | .3935 | 1.00 | 2.7183 | .3679 | .6321 | 3.50 | 33.115 | .0302 | .9698 |
| | | | | .51 | 1.6653 | .6005 | .3995 | 1.05 | 2.8577 | .3499 | .6501 | 3.55 | 34.813 | .0287 | .9713 |
| .21 | 1.2337 | .8106 | .1894 | .52 | 1.6820 | .5945 | .4055 | 1.10 | 3.0042 | .3329 | .6671 | 3.60 | 36.598 | .0273 | .9727 |
| .22 | 1.2461 | .8025 | .1975 | 53 | 1.6989 | .5886 | .4114 | 1.15 | 3.1582 | .3166 | .6834 | 3.65 | 38.475 | .0260 | .9740 |
| .23 | 1.2586 | .7945 | .2055 | | | | | | | | | 3.70 | 40.447 | .0247 | .9753 |
| .24 | 1.2712 | .7866 | .2134 | .74 | 2.0959 | .4771 | .5229 | 2.20 | 9.0250 | .1108 | .8892 | | | | |
| .25 | 1.2840 | .7788 | .2212 | .75 | 2.1170 | .4724 | .5276 | 2.25 | 9.4877 | .1054 | .8946 | 4.75 | 115.58 | .00865 | .99135 |
| .26 | 1.2969 | .7711 | .2289 | .76 | 2.1383 | .4677 | .5323 | 2.30 | 9.9742 | .1003 | .8997 | 4.80 | 121.51 | .00823 | .99177 |
| .27 | 1.3100 | .7634 | .2366 | .77 | 2.1598 | .4630 | .5370 | 2.35 | 10.486 | .0954 | .9046 | 4.85 | 127.74 | .00783 | .99217 |
| .28 | 1.3231 | .7558 | .2442 | .78 | 2.1815 | .4584 | .5416 | 2.40 | 11.023 | .0907 | .9093 | 4.90 | 134.29 | .00745 | .99255 |

TABLE 2. EXPONENTIAL FUNCTIONS.

8.4 Exponential Formulae for Charge and Discharge. By combining all of the processes used in the calculations in paras. 7.2 and 7.3, general formulae can be derived for the voltages and currents in R-C and L-R circuits.

For example, to find a formula for the variation of voltage across the capacitor of an R-C circuit after the applied voltage to the circuit has been altered, consider that the applied voltage is initially $E_A$ and the capacitor is fully charged to this voltage. The input voltage is then changed to $E_B$ so that the capacitor voltage changes exponentially towards $E_B$ as in Fig. 30. The maximum change of voltage ($\Delta E$) is $E_B - E_A$. Thinking of the initial voltage ($E_A$) as the reference voltage, after the change of the input voltage, the capacitor voltage ($e_C$) increases towards a maximum value, and the change as a fraction of the maximum at any point on the curve is described by the equation -

$$y_B = 1 - \epsilon^{-x}$$

As $\Delta E$ is the maximum change, the actual voltage change is -

$$e_C \text{ (change)} = \Delta E \, (1 - \epsilon^{-x})$$

Adding this to the reference voltage, the actual capacitor voltage at any instant is -

$$e_C = E_A + \left[\Delta E \, (1 - \epsilon^{-x})\right]$$

x is the time (t) in time constants -

$$\therefore \; x = \frac{t}{T}$$

$$= \frac{t}{CR} \text{ for R-C circuits.}$$

This formula applies independent of whether $E_A$ or $E_B$ are positive or negative, or the change of voltage is positive or negative.

Other formulae can be derived in a similar manner, and these are given in Table 3. However, in most cases it is more convenient to reason out the shape of the resultant curves, and then use the relevant section of the exponential tables to determine the relationship between time constant and fraction of the maximum value, than to attempt to memorise the formulae of the table.

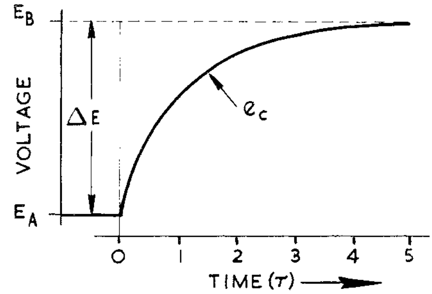



FIG. 30. CAPACITOR VOLTAGE.

| R-C CIRCUITS | | L-R CIRCUITS | |
|---|---|---|---|
| $e_R$ | $= \Delta E \, \epsilon^{-x}$ | $e_R$ | $= E_A + \left[\Delta E \, (1 - \epsilon^{-x})\right]$ |
| $e_C$ | $= E_A + \left[\Delta E \, (1 - \epsilon^{-x})\right]$ | $e_L$ | $= \Delta E \, \epsilon^{-x}$ |
| $i_{(R-C)}$ | $= \frac{\Delta E}{R} \epsilon^{-x}$ | $i_{(L-R)}$ | $= \frac{E_A}{R} + \left[\frac{\Delta E}{R} (1 - \epsilon^{-x})\right]$ |

WHERE
$E_A$ = Initial Voltage
$E_B$ = Final Voltage
$\Delta E$ = Maximum Voltage Change ($E_B$ - $E_A$)

$x = \frac{t}{CR}$ for R-C circuits

$x = \frac{tR}{L}$ for L-R circuits

TABLE 3. EXPONENTIAL FORMULAE FOR R-C AND L-R CIRCUITS.

Example 7. (Using exponential formula.) The capacitor in Fig. 31a has been completely charged with the switch in position 1. The switch is then operated to position 2. What is the capacitor voltage 60mS after the switch operation?

The capacitor is charged to -40 volts ($E_A$). When the switch is in position 2 the voltage attempts to change towards +60 volts ($E_B$) as shown in Fig. 31b. The formula for the capacitor voltage is -

$$e_C = E_A + \left[\Delta E \, (1 - \epsilon^{-x})\right]$$

$$\Delta E = E_B - E_A$$

$$= 60 - (-40) = 100V.$$

$$X \text{ for } 60mS = \frac{t}{CR}$$

$$= \frac{60 \times 10^6}{10^3 \times 2 \times 4 \times 10^4} = 0.75$$

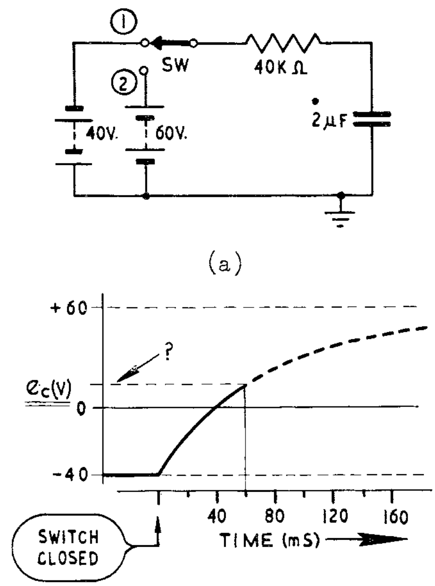
$$\therefore \text{ Voltage after } 60mS = -40 + \left[100 \, (1 - \epsilon^{-0.75})\right]$$

$$= -40 + \left[100 \times .5276\right]$$

$$= -40 + 52.76$$

$$= +12.76 \text{ Volts}$$

Answer:- Capacitor Voltage after 60mS = +12.76 Volts.




(b)

FIG. 31. CIRCUIT FOR EXAMPLE.

## 9. CHARGING FROM A NETWORK.

9.1 In many practical circuits the charge and discharge of capacitors and inductors is from a network of components, valves and voltage sources. It is often required that the change of charge of a capacitor in the anode circuit of a valve as in Fig. 32a be examined when the grid voltage changes abruptly from one value to another. This type of circuit can be considered as being equivalent to a simple voltage divider consisting of the load resistance and the D.C. resistance of the valve for the circuit voltages applying (Fig. 32b). Even so, it is not obvious how the charge of the capacitor will vary or what will be the time constant of the circuit.

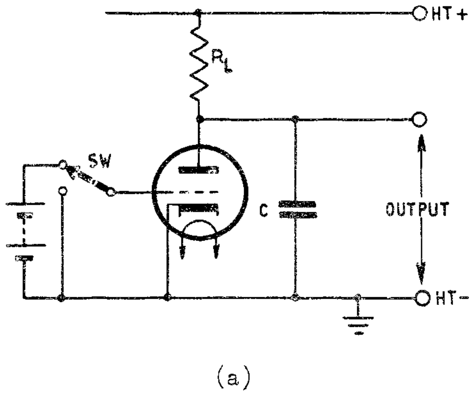



FIG. 32. CAPACITOR CHARGING FROM A NETWORK.

9.2 Thevenin's Theorem. The shape and the timing of the charge curve for the capacitor in Fig. 32, and of capacitors and inductors in other and perhaps more complicated circuits, is easily examined by using Thevenin's Theorem. One way of stating Thevenin's Theorem is -

The current in any impedance connected to a two terminal network of any number of impedances and voltage sources, is the same as when the impedance is connected to a single voltage source equal to the open circuit voltage at the two terminals, and with an internal impedance equal to the impedance measured at the terminals with the voltage source replaced by impedances equal to their respective internal impedances.

This statement is explained in Fig. 33. The network in Fig. 33a supplies current to impedance $Z_L$. With $Z_L$ disconnected the voltage between A and B is $E_O$ (Fig. 33b) and the impedance looking into A and B with any voltage sources in the network replaced by their equivalent impedances, is $Z_O$ (Fig. 33c). The equivalent circuit for the network is then as in Fig. 33d, with a voltage, $E_O$, being applied to terminals A and B via an impedance, $Z_O$.

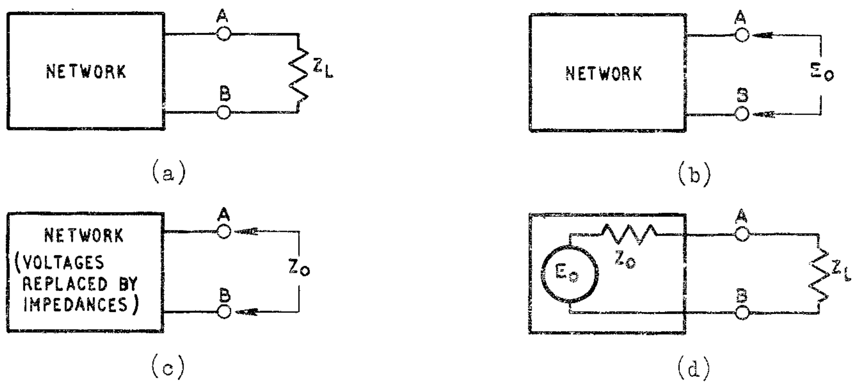



FIG. 33. THEVENIN'S THEOREM.

9.3 A simple problem solved by two methods proves the validity of the theorem and indicates how it is used.

Example 8. In the circuit of Fig. 34, what is the current through $R_3$.

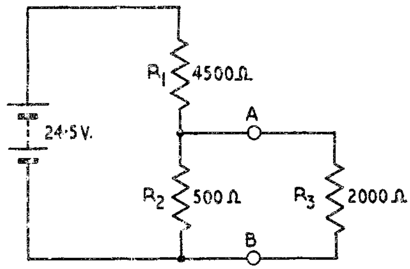



FIG. 34. CIRCUIT FOR EXAMPLE 8.

Solution (i).

$$\text{Parallel resistance of } R_2 \text{ and } R_3 = \frac{500 \times 2000}{500 + 2000} = 400\Omega$$

$$\text{Total Resistance} = 4{,}900\Omega$$

$$\text{Total Current} = \frac{24.5 \times 10^3}{4900} = 5\text{mA.}$$

$$\text{Voltage across } R_2 \text{ and } R_3 \text{ in parallel} = \frac{5 \times 400}{10^3} = 2\text{V.}$$

$$\text{Current through } R_3 = \frac{2 \times 10^3}{2000} = 1\text{mA.}$$

Answer:- Current through $R_3$ = 1mA.

Solution (ii). (Thevenin's Theorem.)

In Fig. 34, with $R_3$ disconnected, the voltage between A and B

$$= \frac{500 \times 24.5}{5000} = 2.45\text{V.}$$

Resistance between A and B with 24.5V battery replaced by a short circuit

$$= \frac{4500 \times 500}{4500 + 500} = 450\Omega$$

The equivalent circuit is shown in Fig. 35.

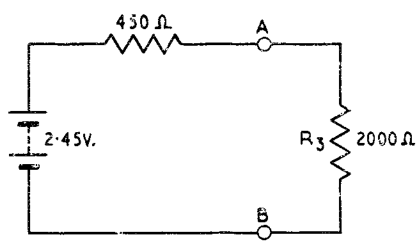



FIG. 35. EQUIVALENT CIRCUIT FOR EXAMPLE 8.

$$\text{From Fig. 35, current through } R_3 = \frac{2.45 \times 10^3}{2450} = 1\text{mA.}$$

Answer:- Current through $R_3$ = 1mA.

9.4 Charging Capacitor from a Complex Source. The method of calculation using Thevenin's Theorem is applied to the charging of a capacitor in the following problem.

Example 9. (i) Find the voltage across the capacitor in Fig. 36a, 210mS after the switch is closed.

(ii) Find the initial charging current through the capacitor.

(i) Voltage across A and B with capacitor disconnected (switch closed) -

$$= \frac{1}{(1 + 0.25)} \times 250 = \frac{250}{1.25} = 200V.$$

Resistance of source with battery replaced by a short circuit -

$$= \frac{(0.25 \times 10^6)(1 \times 10^6)}{(0.25 \times 10^6) + (1 \times 10^6)} = 200k\Omega.$$

∴ Equivalent circuit is as in Fig. 36b.

Time constant of equivalent circuit -

$$= \frac{0.6 \times 0.2 \times 10^6 \times 10^3}{10^6} = 120mS.$$

210mS expressed in time constants -

$$= \frac{210}{120} = 1.75 \text{ time constants.}$$

From exponential functions $(1 - \epsilon^{-x})$, voltage on capacitor after 1.75 time constants -

$= 0.8262$ of maximum

$= 0.8262 \times 200 = 165.24V.$

(ii) The initial charging current from the equivalent circuit -

$$= \frac{200 \times 10^3}{0.2 \times 10^6} = 1mA.$$

Answer:- (i) Capacitor voltage after 210mS = 165.24V.
(ii) Initial charging current = 1mA.

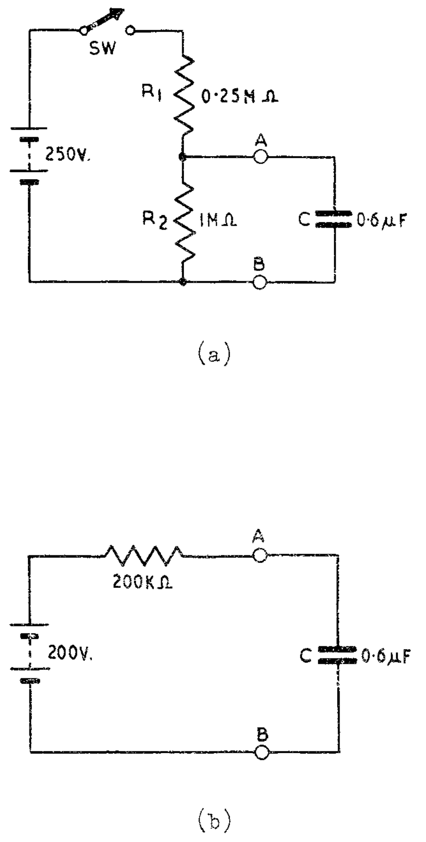



FIG. 36. CIRCUIT FOR EXAMPLE 9.

This initial charging current can be verified by referring to Fig. 36a and considering the voltages when the switch is first closed. With no charge on the capacitor, and therefore no voltage across it, all of the supply voltage is across $R_1$. This means that the initial current through $R_1$ is -

$$i_M = \frac{E_S}{R_1} = \frac{250 \times 10^3}{0.25 \times 10^6} = 1mA.$$

With no voltage across $R_2$ all of this current passes through C as a charging current. This is the same current as calculated from the equivalent circuit.

Also when the capacitor is fully charged, no capacitor current flows, and the voltage across the components is only determined by the voltage divider. The maximum voltage on the capacitor is therefore the voltage across $R_2$ with C effectively disconnected, i.e. $\frac{R_2}{R_1 + R_2} E_S$. This is the same as the maximum voltage on the capacitor in the equivalent circuit, that is the source voltage of the equivalent circuit.

10. TEST QUESTIONS.

1. Draw a rectangular wave and indicate on the wave -

   (i) Pulse Spacing.

   (ii) Pulse Duration.

   (iii) Rise Time.

2. Define "Pulse Duty Factor".

3. Draw approximately the frequency spectrum of a rectangular pulse train with a pulse repetition frequency of 15kc/s and a pulse duration of 5µS. Show the frequency scale and the component frequency spacing.

4. What bandwidth is required to transmit a pulse with a rise time of 0.2µS with negligible degradation.

5. Calculate the peak-to-peak value of a negative rectangular pulse train with a pulse spacing of 50µS and a pulse duty factor of 0.1. The average value of the wave is -5 volts and the maximum positive part of the wave is at -1 volt.

6. The video waveforms on the anode of a video amplifier are shown for a black picture signal in Fig. 37a and for a white picture signal in Fig. 37b. What D.C. anode voltage would be indicated by a moving coil meter for each signal condition?

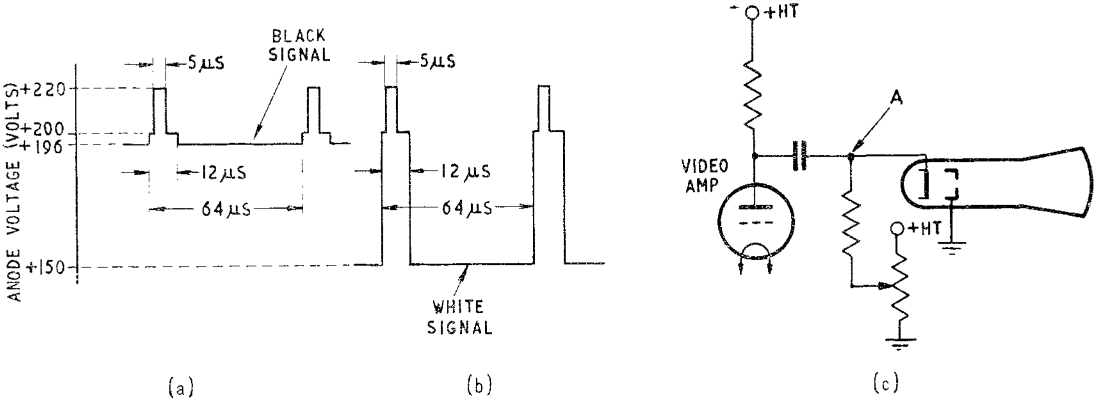



FIG. 37. WAVEFORMS AND CIRCUIT FOR QUESTIONS 6 AND 7.

7. The video signals of Fig. 37 are A.C. coupled to the grid cathode circuit of a television picture tube as in Fig. 37c. Cut-off of the picture tube occurs when the grid is 60V negative with respect to the cathode. The bias is adjusted with each signal in turn so that black level of the signal is at the cut-off voltage. What is the bias voltage required at point A for each signal?

8. (i) Define "time constant" of an R-C circuit.

   (ii) Draw the curves of voltage and current in an R-C circuit when the capacitor is charging from zero to a maximum voltage ($E_M$). Mark the magnitude of each of the curves relative to $E_M$, one time constant after charging has commenced.

9. The time constant of a series R-C circuit is 20mS. A 60V supply is connected for 15mS, removed for 5mS and then connected again for a further 15mS. Calculate the capacitor voltage at the end of this time.

10. Two capacitors, one of 1µF and one of 4µF, are connected in series to a 120 volt supply via a 50kΩ resistor.

    (i) How long does it take the voltage on the

    (a) 1µF capacitor, (b) 4µF capacitor,

    to reach 20 volts?

    (ii) What is the circuit current at the instant the circuit is completed?

11. Fig. 38 is intended to introduce a time delay in relay operation. The switch rests in position 1 and timing starts when the switch is operated to position 2 where a bias voltage of -5 volts is connected to the grid to prevent excess anode current.

The relay operates when the grid is -10 volts and releases when the grid is -15 volts with respect to the cathode in each case.

(i) What value of capacitance is required to give a 4.6 minute delay?

(ii) The switch remains in position 2 until the capacitor is fully charged and then returns to position 1. How long is it before the relay release voltage is reached?

(iii) If the operate voltage required changes to -8 volts, what would be the change in time delay introduced by the circuit?

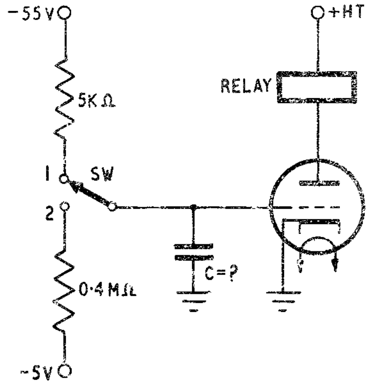



FIG. 38. CIRCUIT FOR QUESTION 11.

12. The switch in Fig. 39 closes for 20mS and then opens again. What is the capacitor voltage 60mS after the switch first closed?

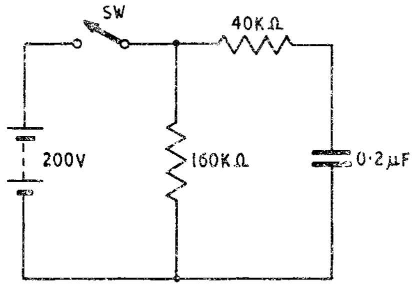



FIG. 39. CIRCUIT FOR QUESTION 12.

13. An amplifier has a coupling circuit consisting of an 0.1μF capacitor and an 0.5MΩ resistor.

(i) What is the time constant of the circuit?

(ii) When the capacitor has a leakage resistance of 1.5MΩ, what is the circuit time constant?

14. A 4 henry inductor with an internal resistance of 400Ω is connected to a battery. The current after 5mS is [illegible]mA.

(i) What is the battery voltage?

(ii) How long will it take for the current through the inductor to reach 20mA when:-

(a) A 400Ω resistor is connected in series with the inductor?

(b) A 400Ω resistor is connected in parallel with the inductor?

(c) A 400Ω resistor is connected in parallel with the inductor and another 400Ω in series with the combination?

15. A 100 volt battery is connected to a 2μF capacitor via a 10kΩ resistor, and the capacitor is completely charged. A 2.5kΩ resistor is then connected across the capacitor.

(i) Draw graphs of the change of:-

(a) voltage across the capacitor;

(b) voltage across the 10kΩ resistor;

(c) current in the capacitor;

(d) current in the 10kΩ resistor;

(e) current in the 2.5kΩ resistor.

(ii) Calculate and indicate on the graphs the initial, maximum and final values.

(iii) How long does it take for the change to be, for practical purposes, completed?

END OF PAPER.